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Edited by

R. P. Boas, Jr. **J. L. Doob** **E. Hille**

J. V. Wehausen, *Executive Editor*

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Mathematical Reviews

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HISTORY

***Becker, Oskar. Grundlagen der Mathematik in geschichtlicher Entwicklung.** Verlag Karl Alber, Freiburg-München, 1954. xi+422 pp. DM 26.00.

This book is designed as a sketch of the historical development of the foundations of mathematics from antiquity to the present, to be read by both mathematicians and laymen. The style is rather unusual and very interesting, consisting of exerpts from original documents and papers connected by brief comments by the author. Both the style and choice of material show a highly individual point of view. Chapter I deals with Egyptian and Babylonian mathematics, Chapter II with the Greek contributions and includes a section of philosophic reflections on the elementary nature and foundations of mathematics. In Chapter III we jump to the 17th century, Galileo, Kepler, and to the discovery of analytic geometry and the invention of calculus. Chapter IV is entitled "The critical mathematics of the 19th century." It is divided into 2 parts. Part I deals with the foundations of geometry and includes work on Euclid's 5th axiom, Kant's theory of space, Gauss's discovery of non-Euclidean geometry, the work of Riemann, Hilbert, Klein, and various philosophical comments. Part II is on the foundations of arithmetic, the work of Bolzano, Dedekind and Cantor. The fifth and last chapter concerns the study of foundations in the 20th century, with exerpts from Frege, Russell, Kronecker, Brouwer, Weyl, Kolmogoroff's interpretation of integers, work of Hilbert, Husserl, Schelling, Gentzen and Lorenzen. Gödel is mentioned twice very briefly in small print, but not his method of the arithmetization of metamathematics, nor is there any mention of recursive functions. Many other important results and methods of modern logic are also absent, as for example Tarski's work on truth theory, which is not mentioned at all. The author thus gives a very personal point of view on the foundations of mathematics. *L. Novak-Gál* (Ithaca, N. Y.).

Guzzo, Augusto. "Posizione" e deduzione in Euclide. Univ. e Politecnico Torino. Rend. Sem. Mat. 13, 1–17 (1954).

Frajese, Attilio. Il contributo personale di Euclide alla costruzione dell'edificio geometrico. Archimede 6, 258–262 (1954).

Bottari, Amerigo. Le tre lunule quadrabili di Ippocrate. Period. Mat. (4) 32, 223–230 (1954).

Karpinski, Louis C. Third supplement to the "Bibliography of mathematical works printed in America through 1850." Scripta Math. 20 (1954), 197–202 (1955).

Boyer, Carl B. Analytic geometry in the Alexandrian age. Scripta Math. 20, 30–36 (1954), 143–154 (1955).

Spasskiĭ, I. G. The origin and history of Russian abacuses. Istor.-Mat. Issled. 5, 267–420 (1952). (Russian)

Gussov, V. V. Works of Russian scholars on the theory of the gamma function. Istor.-Mat. Issled. 5, 421–472 (1952). (Russian)

***Lefschetz, S. Russian contributions to differential equations.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 68–74. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

Finikov, S. P. On the scientific trend of the department of differential geometry of the Moscow State University. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 3–18 (1954). (Russian)

A discussion of research in differential geometry carried out at the Moscow State University from about 1917 to the present. An extensive bibliography accompanies the paper.

Gnedenko, B. V., and Kalužnin [Kaloujnine], L. A. On mathematical life in the German Democratic Republic. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 133–154 (1954). (Russian)

Fihtengol'c, G. M. On transformation of variables in multiple integrals. Istor.-Mat. Issled. 5, 241–268 (1952). (Russian)

Historical paper.

***Dugas, René. De Descartes à Newton par l'école anglaise.** Université de Paris, Paris, 1953. 19 pp.

Brun, Viggo. Niels Henrik Abel. Neue biographische Funde. J. Reine Angew. Math. 193, 239–249 (1954).

***Bernšteĭn, S. N. Sobranie sočineniĭ. Tom II. Konstruktivnaya teoriya funkciĭ [1931–1953]. [Collected works. Vol. II. The constructive theory of functions [1931–1953].]** Izdat. Akad. Nauk SSSR, Moscow, 1954. 627 pp. 34.60 rubles.

Vol. I [1952] was reviewed in these Rev. 14, 2. This volume is arranged in the same way. Several of the more recent papers have been revised and/or extensively annotated by the author, but not to the extent of including references to work done outside the USSR.

R. P. Boas, Jr. (Evanston, Ill.).

Procissi, Angiolo. I "Ragionamenti d'algebra" di R. Canacci. Boll. Un. Mat. Ital. (3) 9, 300–326, 420–451 (1954).

★Carathéodory, Constantin. Gesammelte mathematische Schriften. Bd I. Herausgegeben im Auftrag und mit Unterstützung der Bayerischen Akademie der Wissenschaften. C. H. Beck'sche Verlagsbuchhandlung, München, 1954. xii+426 pp. (1 plate). DM 42.00.

This first volume of the planned five volumes containing the collected works of Carathéodory consists of twenty papers on the calculus of variations, partitioned in subject matter as follows: A) Discontinuous solutions (papers I, II, III); B) General theory (papers IV–XV); C) Multiple integrals (papers XVI–XX). Papers I and II are Carathéodory's Dissertation and Habilitationsschrift, respectively. Paper XI is a hitherto unpublished note on families of extremals written in about 1932, while no. XV is Carathéodory's article on the calculus of variations that forms Chapter V of Frank and von Mises, Die Differential- und Integralgleichungen der Mechanik und Physik [2nd ed., Vieweg, Braunschweig, 1930].

As indicated in the preface written by H. Tietze, throughout the five volumes the papers are to be reproduced in original language, with the exception of translation into German of those papers which were published in Greek; for the present volume the translation of Paper XVII has been performed by D. A. Kappos. In the individual papers alterations have been limited to carefully noted corrections of misprints and errors, with an occasional limited change in wording. In sponsoring this publication the Bavarian Academy of Sciences has performed an outstanding service, especially in view of the great diversity of journals in which the papers of Carathéodory were published originally. It is to be hoped that the four additional volumes will appear in the near future. In particular, reprints of other papers of Carathéodory concerned with the calculus of variations are due to form part of the contents of volumes II and V.

W. T. Reid (Evanston, Ill.).

Finzi, Arrigo. Obituary: Elie Cartan. Riveon Lematematika 8, 76–80 (1954). (Hebrew. English summary)

★Delaunay [Delone], B. N. P. L. Chebichev and the Russian school of mathematics. Academy of Sciences of the USSR, Moscow-Leningrad, 1945. 9 pp.

Segre, Beniamino. L'opera scientifica di Fabio Conforto. Rend. Mat. e Appl. (5) **14**, 48–74 (1954).

Zich, Otakar. On the 410th anniversary of the death of Nicholas Copernicus. Časopis Pěst. Mat. **78**, 297–304 (1953). (Czech)

Court, Nathan Altshiller. Desargues and his strange theorem. II. Scripta Math. **20** (1954), 155–164 (1955).
For part I see same vol. 5–13 (1954); MR 15, 923.

Montel, Paul. Notice nécrologique sur Leonard Eugene Dickson. C. R. Acad. Sci. Paris **239**, 1741–1742 (1954).

Ruffet, J. Obituary: Henri Fehr. Elem. Math. **10**, 1–4 (1955).

Galli, Mario. Sui contributi di Galileo alla fondazione della dinamica. Boll. Un. Mat. Ital. (3) **9**, 289–300 (1954).

Nobile, Vittorio. Sull'argomento galileiano della quarta giornata dei "Dialoghi" e sue attinenze col problema fondamentale della Geodesia. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 426–433 (1954).

Liste des publications de R. H. Germay. Bull. Soc. Roy. Sci. Liège **23**, 340–359 (1954).

Tenca, Luigi. Guido Grandi nelle sue relazioni coi Bolognesi. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **9**, 49–60 (1952).

Truksa, Ladislav. On the sixtieth anniversary of Professor Jaroslav Janko. Časopis Pěst. Mat. **79**, 181–185 (1954). (Czech)

Nádor, György. Kepler's world view and role in the development of the notion of law of nature. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 219–227 (1954). (Hungarian)

Kostovskiĭ, A. N. Aleksandr Sergeevič Kovan'ko (on his sixtieth birthday). Uspehi Mat. Nauk (N.S.) **9**, 2(60), 215–221 (1 plate) (1954). (Russian)

A list of Kovan'ko's published mathematical papers is included.

Biermann, Kurt-R. Über die Untersuchung einer speziellen Frage der Kombinatorik durch G. W. Leibniz. Forschungen und Fortschritte **28**, 357–361 (1954).

This is an examination of one of Leibniz' studies, De numero factuum in tesseris, as yet unpublished, which enumerates the throws of n dice according to the number of appearances of a given face. Proceeding by recurrence, Leibniz shows that binomial coefficients are involved, or as we now say, that the generating function is $(5+t)^n$.

J. Riordan (New York, N. Y.).

J. F. L. Leonardo de Vinci. Gaceta Mat. (1) **6**, 105–107 (1 plate) (1954). (Spanish)

Šostak, R. Ya. Alekseĭ Vasil'evič Letnikov. Istor.-Mat. Issled. **5**, 167–238 (1 plate) (1952). (Russian)

Terracini, Alessandro. Gino Loria (1862–1954). Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88**, 387–392 (1954).

Bouny, F. Quelques remarques historiques et critiques au sujet de la théorie des vecteurs de Massau. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **28**, no. 6, 52–73 (1954).

Academician Dimitrie Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 7–10 (1954). (Romanian)

The works of academician Dimitrie Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 11–17 (1954).

Stoilov, S. Les singularités des fonctions analytiques uniformes et les travaux de l'académicien Dimitrie Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 19–24 (1954). (Romanian. Russian and French summaries)

Vue d'ensemble des travaux de D. Pompeiu sur les singularités des fonctions uniformes d'une variable complexe et, tout particulièrement, des fonctions continues sur leur ensemble singulier, avec indication des principaux développements auxquels ces travaux ont donné lieu.

Résumé de l'auteur.

Osipovskiĭ, T. F. On space and time. Speech delivered in solemn session of Har'kov University, 30 August 1807, by Professor Timofeĭ Osipovskiĭ. Istor.-Mat. Issled. **5**, 9–17 (1 plate) (1952). (Russian)

Osipovskiĭ, T. F. On Kant's dynamical system. Discourse delivered in solemn session of the Har'kov University, 30 August 1813, by Professor T. F. Osipovskiĭ. Istor.-Mat. Issled. **5**, 18–27 (1952). (Russian)

Bahmutskaya, È. Ya. Timofeĭ Fedorovič Osipovskiĭ and his "Course of mathematics". Istor.-Mat. Issled. **5**, 28–74 (1952). (Russian)

Prudnikov, V. E. Supplementary information on T. F. Osipovskiĭ. Istor.-Mat. Issled. **5**, 75–83 (1952). (Russian)

Peterson, Sven R. Benjamin Peirce: mathematician and philosopher. J. Hist. Ideas **16**, 89–112 (1955).

Quintas Castañs, V. Jacobo Rodriguez Pereira, a Spanish precursor? Calc. Automat. Cibernet. **3**, no. 8, 29–32 (1954).

Peterson, Karl Mihaĭlovič. On the bending of surfaces. Dissertation of K. Peterson to obtain the degree of candidate. Istor.-Mat. Issled. **5**, 87–112 (2 plates) (1952). (Russian)

Translation of the author's previously unpublished "candidate's" thesis, "Über die Biegung der Flächen" [Dorpat, 1853].

Rossinskiĭ, S. D. Commentary on the dissertation of K. M. Peterson, "On the bending of surfaces." Istor.-Mat. Issled. **5**, 113–133 (1952). (Russian)

Depman, I. Ya. Karl Mihaĭlovič Peterson and his candidate's dissertation. Istor.-Mat. Issled. **5**, 134–164 (1952). (Russian)

Have

***Œuvres de Henri Poincaré.** Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome IX. Publié avec la collaboration de Gérard Petiau. Gauthier-Villars, Paris, 1954. xvi+704 pp. (1 plate).

This volume contains Poincaré's papers on mathematical physics and on physical theories. As in earlier volumes, this one also contains relevant parts of Poincaré's Analyse de ses travaux scientifiques [Acta Math. **38**, 3–135 (1921), pp. 116–125]. There is also reproduced an essay by Hadamard [ibid. **38**, 203–287 (1921)] on Poincaré's work and one by H. A. Lorentz [ibid. **38**, 293–308 (1921)] on two of Poincaré's memoirs on mathematical physics. There is a preface by L. de Broglie and notes on individual papers by Petiau. [For earlier volumes see MR **13**, 421, 810; **15**, 227.]

Have

***Selected papers in statistics and probability by Abraham Wald.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. ix+702 pp. (1 plate). $8.00.

Reproduction by photo-offset of fifty-one selected papers by Wald (some jointly with others). There is also a brief biography, a discussion of the papers, and a complete list of Wald's published works.

FOUNDATIONS

Have

***Hermes, H., und Scholz, H. Mathematische Logik.** Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. II, 1. Band I. Algebra und Zahlentheorie. 1. Teil. A. Grundlagen. B. Algebra. Heft 1, Teil I. B. G. Teubner, Leipzig, 1952. 82 pp. 8.20 DM.

This monograph treats the propositional calculus, the predicate calculus of first order with and without identity, but excluding Herbrand's theorem, the theory of effective decision methods, the incompleteness theorems of Gödel and completeness theorems of Henkin for predicate calculi of higher order. [A very detailed review has appeared in J. Symbolic Logic **19**, 278–282 (1954).] The technical parts are clear and concise; the authors' "semantic" point of view is described too briefly to be very coherent.

The authors state (i) that mathematical logic is intended to provide a precise(r) formulation of the notion of consequence on which mathematical theories are based, (ii) that they consider those mathematical theories which fall within the scope of two-valued logic 'where every proposition is true or false'. There is no clear indication which branches of mathematics satisfy this condition; it seems that abstract algebras and other axiomatic theories do, intuitive geometry and numerical arithmetic, called a 'kind' of calculus on p. 48, apparently do not. Nor is it clear why, on their criterion, an inconsistent logic is to be rejected since this may be a 'correct' formulation of the (inconsistent) mathematical theory considered.

Great stress is laid on semantic results, which are formulated in terms of arbitrary mappings and arbitrary propositions (the latter notion is not problematical since all propositions treated here can be formulated in the theory of types). However, practically all substantial results are obtained by means of syntactic arguments, whose content is then diluted in a semantic statement; e.g. the proof on pp. 41–44 does not only establish the existence of a model for every consistent formula of the predicate calculus, but provides information about the notation needed to define the model and the methods needed for a formal proof of the fact that the definitions yield a model. If, as appears, this additional information is of secondary interest (from the semantic point of view), the algebraic treatment of the predicate calculus by Rasiowa and Sikorski [e.g. Fund. Math. **40**, 62–95 (1953); MR **15**, 668] which short-circuits syntactic details, would have been more suitable here.

Reviewer's note: 9(a) on p. 28 can be sharpened. If a formula of the predicate calculus with identity is valid in a denumerable domain it is valid in all domains with cardinal exceeding some finite n. *G. Kreisel* (Reading).

Have

***Beth, E. W. Les fondements logiques des mathématiques. 2ème éd.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1955. xv+241 pp. 2500 francs.

Although there are many changes in detail the general contents and plan are as in the first edition [1950; MR **12**, 71].

Ishimoto, Arata. A set of axioms of the modal propositional calculus equivalent to S3. Science of Thought **1**, 1–11 (1954).

Ishimoto considers a modal propositional calculus with two primitive functions which he denotes by / and //. The first of these is the incompatibility connective of Sheffer and the second denotes a certain kind of logical incompatibility. He defines material implication, negation, conjunction and disjunction in terms of / in the usual way and he defines strict implication and possibility by "$p \supset q$ for

$p//(q/q)$", "$\Diamond p$ for $(p//p)/(p//p)$" respectively. Apparently lower case Roman letters are used for syntactical variables.

He then sets up a formalisation of the system using the rules of adjunction and modus ponens and axioms of the following forms:

$$p//(q/r)\mathrel{\overline{\supset}}(s/q\mathrel{\overline{\supset}}p/s),\ p\mathrel{\overline{\supset}}p,\ p//(q/r)\mathrel{\overline{\supset}}(s//q\mathrel{\overline{\supset}}p//s),$$
$$p//q\mathrel{\overline{\supset}}p/q,\ p//p\mathrel{\overline{\supset}}p//q,$$
$$((p/q)/(p/q))//((p/q)/(p/q))\mathrel{\overline{\supset}}p//q.$$

He derives the axioms and rules of procedure of the Lewis system S3 and also shows that the Lewis definitions of disjunction, material implication and strict implication are provable as the respective equivalences in his system, thus showing that any elementary proposition which is provable in S3 is also provable in the present system. Conversely, it is shown that if $p//q$, p/q are defined to be $\sim\Diamond(p\wedge q)$, $\sim(p\wedge q)$ respectively, then all the provable elementary propositions of the present system are provable in S3 as are Ishimoto's definitions when expressed as strict equivalences.

A. Rose (Nottingham).

Seki, Setsuya. A metatheorem on *SLK*. Comment. Math. Univ. St. Paul. **3**, 31–36 (1954).

LSK is a system obtained by adding to the first-order predicate calculus (*LK*) a hierarchy of types, but not axioms of abstraction. If a formula of *LK* can be proved in *LSK*, then it can also be proved in *LK*. This is a trivial consequence of the fact that any theorem of *LSK* can be proved without a cut. *G. Kreisel* (Reading).

Trahtenbrot, B. A. On recursive separability. Dokl. Akad. Nauk SSSR (N.S.) **88**, 953–956 (1953). (Russian)

It is well known that there is an analogy between the theory of recursive functions and the theory of analytic sets; further that this analogy breaks down in that it is possible to have two disjoint recursively enumerable sets which cannot be separated by a recursive set. [See Mostowski, Fund. Math. **34**, 81–112 (1947); MR **9**, 129; Kleene, Nederl. Akad. Wetensch. Proc. **53**, 800–802 (1950); MR **12**, 71.] The author shows that the set of all identically true formulas of the first-order predicate calculus and the set of all formulas refutable in some finite domain give rise (by a Gödel enumeration) to such a pair of recursively inseparable sets. For a more detailed review see J. Symbolic Logic **19**, 60 (1954). *H. B. Curry.*

Markov, A. A. On the continuity of constructive functions. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 226–230 (1954). (Russian)

The author gives definitions of recursive rational function, recursive convergence, calculable real number, constructive sequence of such numbers, constructive convergence, constructive real function, and related notions. In these he makes use of a principle akin to the Kleene normal-form theorem for recursive functions; viz. that there is a recursive rational function Φ_n of $n+1$ variables such that for every recursive rational function ϕ_n of n variables there can be constructed a number e such that

$$\phi(x_1, \cdots, x_n)=\Phi_n(e, x_1, \cdots, x_n)$$

when either function is defined. He then states theorems such as the following. A constructive real function cannot have a constructive discontinuity; a sequence having a limit point is converted by a constructive function into another sequence having a limit point at the corresponding point; it is not possible to find a constructive function which is defined over an interval and takes on exactly two values; a constructive function defined over an interval takes on every constructive value between any two of its constructive values; and a constructive function defined over an interval (A, B), and such that its values at A and B belong to that interval, has at least one invariant point. *H. B. Curry.*

Have

★Markov, A. A. The theory of algorithms. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 191–203. Akadémiai Kiadó, Budapest, 1952. (Russian. Hungarian summary)

Essentially the same material is contained in Trudy Mat. Inst. Steklov. **38**, 176–189 (1951); MR **13**, 811.

Nagornyĭ, N. M. On strengthening the reduction theorem of the theory of algorithms. Dokl. Akad. Nauk SSSR (N.S.) **90**, 341–342 (1953). (Russian)

Markov [see the paper above and reference there] stated that to every normal algorithm $\mathfrak{A}$ over an alphabet A (i.e. converting a word of A into another word of A, but using in its scheme of rules auxiliary symbols not in A) there is a normal algorithm $\mathfrak{B}$ having the same effect on words of A and using only two auxiliary symbols. The present author shows that a $\mathfrak{B}$ can be constructed using only one auxiliary symbol; but a $\mathfrak{B}$ cannot always be constructed without any auxiliary symbols. *H. B. Curry.*

Detlovs, V. K. Normal algorithms and recursive functions. Dokl. Akad. Nauk SSSR (N.S.) **90**, 723–725 (1953). (Russian)

The author asserts the equivalence of Markov's notion of normal algorithm [see the paper second above and reference there] with the notion of recursive numerical function. More precisely he takes an alphabet C of two symbols '|' and '$*$', in which he represents the integer n by a set of n strokes and an ordered set of m numbers $x_1, \cdots, x_m$ by $x_1*\cdots*x_m$, he then calls a function $\phi(x_1, \cdots, x_n)$ algorithmic if its value, when it exists, can be obtained (as a series of strokes) by a normal algorithm over C from the initial word $x_1*\cdots*x_m$; it is completely algorithmic if such a value can be obtained for arbitrary $x_1, \cdots, x_m$. Then every partial recursive function is algorithmic and vice versa; likewise every completely algorithmic function is general recursive and vice versa. *H. B. Curry.*

Specker, E. Verallgemeinerte Kontinuumshypothese und Auswahlaxiom. Arch. Math. **5**, 332–337 (1954).

Let Σ be the system of axioms for set theory as formulated by Bernays, but without the axiom of choice or the Fundierungsaxiom. The following definitions and proofs can then be formalized in Σ: A class C is a cardinal if C has as its elements all sets d which can be mapped one-to-one onto every element of C. If M is a cardinal, and d is in M then 2^M is the cardinal of the powerclass of d. A cardinal M is called an Aleph, if it contains an ordinal. If M is an Aleph, then every set $d\in M$ can be well ordered, since it can be put into one-to-one correspondence with the ordinal in M. The cardinal M is said to be less than or equal to the cardinal N ($M\leqq N$), if there is a set $c\in M$ and a set $d\in N$ such that $c\subseteq d$. For every cardinal M there is a least Aleph $\aleph(M)$ such that $\aleph(M)$ is not less than or equal to M ($\aleph(M)$ not $\leqq M$). A cardinal M is said to fulfill the continuum hypothesis $H(M)$, if whenever X is a cardinal such that $M\leqq X\leqq 2^M$, then $X=M$ or $X=2^M$. Specker then proves the following generalizations of theorems of Lindenbaum and Tarski and

Sierpiński (without of course using the axiom of choice): If $H(M)$ and $H(2^M)$, then 2^M is an Aleph (and therefore can be well ordered). *L. Novak Gál* (Ithaca, N. Y.).

Martin, Norman M. Note on the completeness of decision element sets. J. Computing Systems 1, 220 (1954).

Guggenheimer, H. Quelques remarques concernant l'article de M. Alexandre Wittenberg: Über adäquate Problemstellung in der mathematischen Grundlagenforschung. Dialectica 8, 145–146 (1954).

Bernays, P. Bemerkungen zu der Betrachtung von Alexander Wittenberg: Über adäquate Problemstellung in der mathematischen Grundlagenforschung. Dialectica 8, 147–151 (1954).

Wittenberg, Alexander. Über adäquate Problemstellung in der mathematischen Grundlagenforschung. Eine Antwort. Dialectica 8, 152–157 (1954).

[See Wittenberg, Dialectica 7, 232–254 (1953); MR 15, 593.] According to Guggenheimer logic is a part of mathematics and the latter is an experimental science. Bernays stresses that "the free and conscient comparison of the contrasting trends agrees exactly with the attitude of scientific objectivity." He hopes that a satisfactory revised form of platonicism can be found. Wittenberg answers that the simultaneous admission of contrasting trends, as well as a more or less arbitrary revision of platonicism, is incompatible with the claim for a realm of objectivity. This is why he proposed to secure objectivity by epistemologic research. *A. Heyting* (Amsterdam).

have

***Sublet, Jacques. Essai de formalisation complète du raisonnement mathématique sur la base de trois opérations.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 91–94. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2200 francs.

Esquisse très brève. Les trois opérations correspondent aux signes ϵ (signification usuelle), | (barre de Sheffer) et $\rho_x A$ (un élément quelconque x satisfaisant A).

A. Heyting (Amsterdam).

Vaccarino, Giuseppe. L'origine delle classi. Methodos 6, 5–36 (1954).

have

***Destouches, Jean-Louis. La logique et les théories physiques.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 119–126; discussion pp. 126–128. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

The author points out the roles of deduction, verification, formal structure, measure, and prediction in the development of physical theory. *C. C. Torrance.*

have

***Destouches-Février, P. La logique des propositions expérimentales.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 115–118. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

An enumeration of seven languages that must be developed in succession in order to deal with physical theory.

C. C. Torrance (Monterey, Calif.).

Pauli, W. Wahrscheinlichkeit und Physik. Dialectica 8, 112–118; Diskussion 118–124 (1954).

The author begins by pointing out that an aim of mathematical probability is to make subjective expectations as objective as possible. [Cf., e.g., the reviewer's Probability and the weighing of evidence, Griffin, London, 1950, p. 4; MR 12, 837.] As an example it is shown that Bernouilli's theorem cannot be applied in practice without the use of judgment. It is then argued that even when applying in practice the probabilities that occur in physical theories one must rely on (subjective) judgment. The author recalls that in wave mechanics the interpretation of wave functions is in terms of probabilities, but that the laws obeyed by the wave functions are deterministic, and he suggests that this unifies some apparently contradictory views of Parminedes and Heraclitus. There is an animated discussion by Destouches, Bernays and Gonseth. *I. J. Good.*

Mahalanobis, P. C. The foundations of statistics. Dialectica 8, 95–111 (1954).

The author argues that while "it is always difficult to be sure about the exact meaning of logical and philosophical phrases which were current 1500 or 2500 years ago", nevertheless the Jaina logic called syādvāda has some relevance to the concepts of probability and statistics. In the reviewer's opinion the relevance is not great.

For any entity, let F, G, H respectively mean that the entity is, that it is not, and that it is indescribable or indeterminate. Then, according to syādvāda, the following $2^3-1=7$ propositions are mutually exclusive and exhaustive, and each has non-zero probability: F, G, $F\&G$, H, $F\&H$, $G\&H$, $F\&G\&H$ (the "sevenfold categories of knowledge"). The apparently self-contradictory proposition $F\&G$ probably means that in some sense the entity exists, while in some other sense it does not exist. The notion of probability is only implicit and qualitative and is expressed by the word syāt ("maybe").

According to Jaina logic no statement is absolutely true, a view that is at least analogous to the one that no empirical proposition has probability 1.

The author is concerned only with Jaina logic, not with its ethics; otherwise he might have mentioned that the modern statistical practice of dosage mortality tests is inconsistent with the Jaina insistence on the avoidance of taking any form of life. *I. J. Good* (Cheltenham).

Sur un aspect paradoxal de la théorie des probabilités. Discussion mise au point par Guy Hirsch. Dialectica 8, 125–144 (1954).

If in a sequence of n causally independent 'trials' there are m 'successes', it is often desired to make a statement of the form $|p-m/n|<\epsilon$, where p is the 'chance' of a success. Such a statement cannot be made except with some probability of error, and there is a well-known danger of running into an infinite regress in the attempt to make a definite statement. This 'paradox' is essentially the same as the circularity of the limiting frequency definition of probability. [See, e.g., the reviewer's Probability and the weighing of evidence, Griffin, London, 1950, p. 6; MR 12, 837.] In order to avoid the infinite regress the author suggests a 'dialectical' attitude, in which all statements are regarded as subject to revision. Gonseth is acknowledged for this dialectical attitude in science as a whole, though he seems to have been foreshadowed by the originators of syādvāda and by Hegel [see the preceding review]. The reviewer

thinks that the dialectical attitude need not be invoked if use is made of subjective probabilities or of 'credibilities'. There is a detailed discussion by Destouches, Bernays, Richter, Ostrowski, Fierz, Feraud, Nolfi and Linder.

I. J. Good (Cheltenham).

Tarski, Alfred. The semantic conception of truth and the foundations of semantics. Euclides, Groningen **30**, 1–43 (1955). (Dutch)

Translation by E. W. Beth of a paper in Philos. and Phenomenol. Res. **4**, 341–376 (1944); MR **6**, 31.

Have ***Weise, Karl-Heinrich. Vom mathematischen Denken.** Ferdinand Hirt, Kiel, 1953. 30 pp. DM 1.20.

***Montel, Paul. La recherche scientifique en mathématiques.** Université de Paris, 1952. 23 pp.

Köthe, Gottfried. On the noncontradictoriness of mathematics. Gaz. Mat., Lisboa **15**, no. 58, 1–5 (1954). (Portuguese)

Expository lecture.

Carruccio, E. La logica matematica nel passato e nel presente della scienza. Scientia (6) **89**, 317–324 (1954).

***Ridder, J. Aard en structuur der wiskunde. [The nature and structure of mathematics.]** P. Noordhoff N. V., Groningen-Djakarta, 1950. 24 pp. 0.90 florins.

Lecture given at the University of Groningen in May, 1950.

***Heyting, A. Spanningen in de wiskunde. [Stresses in mathematics.]** P. Noordhoff N. V., Groningen-Batavia, 1949. 20 pp. 0.90 florins.

Lecture given at the University of Amsterdam in May, 1949.

***Gerretsen, J. C. H. De betekenis van de wiskunde voor de hedendaagse natuurwetenschap. [The meaning of mathematics for contemporary science.]** P. Noordhoff, Groningen-Batavia, 1949. 27 pp. 1.25 florins.

Lecture given at Groningen in April, 1949.

***Zaanen, A. C. Enige motieven die bij de beoefening der wiskunde ook een rol spelen. [Some motives which play a role in the study of mathematics.]** P. Noordhoff N. V., Groningen-Djakarta, 1951. 16 pp. 1.25 florins.

Lecture given at the Technische Hogeschool te Delft, January, 1951.

***Kloosterman, H. D. Waarde en waardeering der wiskunde. [The value and appreciation of mathematics.]** P. Noordhoff N. V., Groningen-Batavia, 1947. 16 pp. 0.90 florins.

Lecture given at the University of Leiden in May, 1947.

Zaragüeta Bengoechea, Juan. Problematics of the philosophy of science. Mem. Real Acad. Ci. Art. Barcelona **31**, 365–384 (1954). (Spanish)

ALGEBRA

***Gantmaher, F. R. Teoriya matric. [The theory of matrices.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 491 pp. 20.90 rubles.

The work under review is a text and treatise on the theory of matrices with real or complex elements, with selected applications to problems usually classified in other fields. The first part (10 chapters; "General theory") gives in satisfactory detail, and with more than customary completeness, the topics which belong to the main body of the following subjects: vector spaces and linear operators; elimination in linear systems; minimal, characteristic polynomials; elementary divisors; analytic functions of matrices, canonical forms; $AX-XB=C$; unitary, euclidean spaces; quadratic, Hermitian forms. Numerous correlative subjects are also treated. The point of view is broad, and includes much abstract treatment. Schur's Theorem [Math. Ann. **66**, 488–510 (1909)] that every matrix has a unitary transform in triangular form is not used; for many proofs, the Hamilton-Cayley Theorem takes its place. Examples of theorems proved in the first part of the book are the following. 1. Let A, B be real similar normal matrices ($AA^*=A^*A$, $BB^*=B^*B$, $T^{-1}AT=B$). Then there is a real orthogonal transformation carrying A into B. 2. A real quadratic form is positive definite if the leading principal minors are positive. Numerical examples are worked out, including calculation of characteristic polynomials by a special method [K. A. Semendyaev, Akad. Nauk SSSR. Prikl. Mat. Meh. **7**, 193–221 (1943); MR **6**, 51; **13**, 691; A. M. Lopšic, Trudy Sem. Vektor. Tenzor. Anal. **7**, 233–259 (1949); MR **13**, 991; **14**, 412]. The book has no exercises for the reader.

The second part is devoted to "special topics and applications." The chapter titles are: 11. Complex symmetric, anti-symmetric, and orthogonal matrices; 12. Singular pencils of matrices; 13. Matrices with non-negative elements; 14. Applications of matrices to analyzing systems of differential equations; 15. The Routh-Hurwitz (stability) theorem and related questions. These subjects are treated in a very thorough fashion, and the proofs are snappy. Many theorems and lemmas extend more widely known results. For example: If G is a matrix such that $G=G^*$, $G'G=E$ then there exist J, K, $J=\bar{J}=J'$, $K=\bar{K}=-K'$ with $J^2=E$, $G=Je^{iK}$. (The further condition that G be positive definite would have forced the relations $J=I$, $G=e^{iK}$.) Again: an anti-symmetric matrix always has even rank (more widely known: an anti-symmetric matrix of odd order is not regular). Chapter 13 contains applications of its subject matter to the study of Markoff processes and also to the study of small oscillations of elastic systems. The definitions are clear and the results expounded appear with satisfying generality; for example: "Limiting probability for homogeneous Markov chains with a finite number of states." Chapter 14 contains sections on systems of linear differential equations with variable (not constant) coefficients, Lyapunov's transformation, and canonical reductions at the beginning; the last section of the chapter is concerned with analytic functions of several matrices. Chapter 15 contains a unified treatment of the Routh theory; the chapter tours through a wide variety of related subjects, including infinite circulants, and ends with a generalization of the Routh-Hurwitz problem to the case of complex coefficients.

The number of subjects which the book treats well is great. Since many of these subjects are in fields commonly classified as applied mathematics, an early translation of the book would appeal to a wide audience.
J. L. Brenner (Aberdeen, Md.).

Charles, B. Sur l'algèbre des opérateurs linéaires. J. Math. Pures Appl. (9) **33**, 81–145 (1954).

The main results in this paper were announced in five notes [C. R. Acad. Sci. Paris **236**, 990, 1122–1123, 1722–1723, 1835–1837 and 2027–2029 (1953); MR **14**, 768, 721, 939]. A bibliographical note may now be added. The theorem on the double commuter of a linear transformation announced in the second of these notes (Theorem 33 of the present paper) also appears in the reviewer's Infinite abelian groups [MR **16**, 444]. *I. Kaplansky* (Chicago, Ill.).

Ribeiro, Hugo. A classroom note on the proof of Schur's lemma. Gaz. Mat., Lisboa **15**, no. 58, 11 (1954).

Mardešić, Sibe. Über die Unabhängigkeit mod (G) **der ganzzahligen Linearformen.** Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 280–292 (1953). (Serbo-Croatian summary)

Soit G un groupe abélian additif, X un ensemble, $G(X)$ le groupe additif des chaines sur X, c'est à dire des combinaisons linéaires formelles finies d'éléments de X a coefficients dans G. Si $U\subset Z(X)$, et si $\mathcal{G}$ désigne une classe de groupes abeliens additifs, l'auteur dit que U est indépendant mod $\mathcal{G}$ lorsque, quel que soit $G\in\mathcal{G}$, U est un ensemble d'éléments indépendants dans $Z(X)$ considéré avec G comme domaine d'opérateurs. Il définit une relation d'ordre entre ces diverses notions d'indépendance en écrivant $\mathcal{G}_1\leqq\mathcal{G}_2$ lorsque tout U indépendant mod $\mathcal{G}_2$ est indépendant mod $\mathcal{G}_1$; il écrit $\mathcal{G}_1\approx\mathcal{G}_2$ lorsque $\mathcal{G}_1\leqq\mathcal{G}_2$ et $\mathcal{G}_2\leqq\mathcal{G}_1$. Il désigne par $p(\mathcal{G})$ l'ensemble des nombres premiers m tels qu'il existe $G\in\mathcal{G}$ et $g\in G$, $g\neq 0$, tels que $mg=0$. Une suite de propositions, de démonstrations faciles, lui permet d'obtenir l'énoncé suivant (Th. 6): Pour que $\mathcal{G}_1\leqq\mathcal{G}_2$, il faut et il suffit que $p(\mathcal{G}_2)\subset p(\mathcal{G}_1)$. Il en résulte aisément que lorsque $p(g)$ est vide, $\mathcal{G}\approx\{Z\}$ et que, dans le cas contraire, $\mathcal{G}\approx\{(Z/(m))\mid m\in p(\mathcal{G})\}$. *J. Riguet.*

Ninot, Joachim. Über den Hauptsatz der Galoisschen Theorie. (Kommutative Körper.) Arch. Math. **6**, 52–54 (1954).

This is a brief and neat exposition of the Fundamental Theorem of Galois Theory, based on Artin's definition of a normal extension [Galois theory, 2nd ed., Univ. of Notre Dame, 1944, p. 41; MR **5**, 225]. The only tools used are those of linear algebra and the duality between sets of automorphisms of a field and the subfields which they leave invariant. *W. Ledermann* (Manchester).

Abstract Algebra

Balachandran, V. K. A characterization for complete Boolean algebras. J. Madras Univ. Sect. B. **24**, 273–278 (1954).

Given a lattice L with 0 and I, denote by π_a the set of elements of L with product-complement 0, and dually for π_s. Theorem: If $\pi_a=0_a$ (the zero of the lattice L_a of all dual ideals of L) and L is complete and distributive, then every element of L has a sum-complement. Theorem: A complete distributive lattice L is a complete Boolean algebra if and only if π_a and π_s are zeros of L_a and its dual respectively. In the latter theorem, neither "distributive" nor "complete" can be omitted. The results are obtained through analysis of the composition of π_a and of its relationships to the last residue class and cut-complements. *P. M. Whitman.*

Copeland, A. H., Sr., and Harary, Frank. A characterization of implicative Boolean rings. Canad. J. Math. **5**, 465–469 (1953).

Durch eine interessante Beweisführung wird für die vom Verf. eingeführten implikativen Booleschen Ringe (diese besitzen ausser Addition und Multiplikation noch ein $\times$-Produkt) bewiesen, dass in einem Booleschen Ring B genau dann ein solches $\times$-Produkt definiert werden kann, wenn B für jedes Element $a\neq 1$ isomorph zu $B/(a)$ ist. Diese Isomorphismen liefern dann eine 1-parametrige Transformationshalbgruppe und das Produkt der Transformationen definiert das $\times$-Produkt der Parameter.
P. Lorenzen (Bonn).

Łoś, J., and Ryll-Nardzewski, C. Effectiveness of the representation theory for Boolean algebras. Fund. Math. **41**, 49–56 (1954).

The authors discuss the extent to which the assumption of the existence of prime ideals in Boolean algebras is equivalent to the axiom of choice. They obtain a number of conditions equivalent to the prime-ideal assumption, one of which is the restricted Tychonoff theorem, to the effect that the product of compact Hausdorff spaces is compact. It is known that the general Tychonoff theorem is equivalent to the axiom of choice [cf. J. L. Kelley, Fund. Math. **37**, 75–76 (1950); MR **12**, 626]. *L. H. Loomis.*

Vilhelm, Václav. Theorem of Jordan-Hölder in lattices without finite chain condition. Čehoslovack. Mat. Ž. **4**(79), 29–49 (1954). (Russian. English summary)

The analysis of Kořínek [Rozpravy II. Třídy České Akad. **59**, no. 23 (1949); Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **50**, 307–324 (1951); MR **12**, 667; **13**, 525; the terminology of these references is adopted] is generalized by weakening the finite chain condition. Let S be a lattice in which (I) each quotient, viewed as a sublattice of S, is a complete lattice; (II) for any ascending chain $\{a_\iota\}$, $0\leqq\iota\leqq\rho$, in S, any $c\in S$, and any $\lambda\neq 0$, $\bigvee_{\iota<\lambda}(a_\iota\cap c)=(\bigvee_{\iota<\lambda}a_\iota)\cap c$; (III) if $\{a_\iota\}$ is maximal, a_α, $a_\beta\in\{a_\iota\}$, and $a_\alpha<a_\beta$, then there exist a_κ, $a_\lambda\in\{a_\iota\}$ with $a_\alpha\leqq a_\kappa<a_\lambda\leqq a_\beta$ and a_λ covers a_κ in S. Theorem: Then the Jordan-Hölder theorem with respect to γ holds in S if and only if (1) S satisfies the lower prime quotient condition, and (2) for each maximal irreducible ascending chain $\{a_\iota\}$, $0\leqq\iota\leqq\rho$, and each prime quotient a_ρ/d, $\lambda\neq\rho$ implies $\bigwedge_{\lambda<\iota}(a_\iota\cup d)=(\bigwedge_{\lambda<\iota}a_\iota)\cup d$. And then the Jordan-Hölder relationship holds through the Zassenhaus refinement. It is observed that the lattice of subgroups of a group satisfies I, II, III, as does a lattice in which every non-void subset of a quotient has at least one maximal element. If the lower prime quotient condition holds in S, and there is a finite maximal chain between a and b, then all other maximal chains are finite and of the same length (repetitions omitted), and are related by the Zassenhaus construction. The effect of replacing the lower prime quotient condition by a Birkhoff condition is studied.
P. M. Whitman (Silver Spring, Md.).

Jakubík, J., and Kolibiar, M. On some properties of a pair of lattices. Čehoslovack. Mat. Ž. **4**(79), 1–27 (1954). (Russian. English summary)

Given a lattice S, a partition of the elements of S is called determining if it is a congruence relation on the lattice. Define

$$(a, b, c)=(a\cup b)\cap(b\cup c)\cap(c\cup a).$$

Then $(a, t, x)=(a, t, y)$ defines a congruence relation $x\equiv y$ if S is distributive. If also S has O and I, and t has a complement, the determining partition given by that congruence relation is called the principal determining partition. Consider two lattices S_1, S_2, defined on the same set M, with operations and relations $\cap$, $\cup$, $\subseteq$ in S_1 and $\wedge$, $\vee$, $\leqq$ in S_2. The relations between the following possible properties of such pairs are studied. (A) Every partition of M which is determining on S_1 is determining on S_2 and vice versa. (B) If subset X of M forms a convex sublattice of S_1, so is it a convex sublattice of S_2, and vice versa. (C) $\cap$ and $\cup$ are distributive with both $\wedge$ and $\vee$, and vice versa. (D) There exist lattices A, B for which the following isomorphisms of direct products hold: $S_1\simeq A\times B$, $S_2\simeq A'\times B$, where A' is the dual of A. The following properties are defined only in distributive lattices with O and I. (E) There exist $t\in S_1$ and its complement t' with $x\vee y=(x, t, y)$, $x\wedge y=(x, t', y)$ for all x, $y\in M$. (A_1) Every principal determining partition on S_1 is a principal determining partition on S_2 and vice versa. For M finite, consider: (F) The unoriented graphs of S_1, S_2 are isomorphic. Theorems: In distributive lattices, (A), (B), (C), and (D) are equivalent; if O and I exist, all except (F) are equivalent; if also S_1, S_2 are finite, all are equivalent. In general lattices, (C) implies (B); (D) implies (A). In finite lattices, (D) implies (F). Some counterexamples are given. While proving the theorems, considerable detailed information is recorded on the behavior of the entities involved, such as partitions.

P. M. Whitman (Silver Spring, Md.).

Jakubík, Ján. On lattices whose graphs are isomorphic. Čehoslovack. Mat. Ž. **4**(79), 131–141 (1954). (Russian. English summary)

A lattice is called discrete if every chain with greatest and least elements is finite. Theorem 1: If S_1, S_2 are finite (or discrete) modular lattices, then condition (D) [see the preceding review] is equivalent to (F). Theorem 2: If S_1 is a finite (or discrete) modular lattice, then for every (every discrete) modular lattice S whose unoriented graph is isomorphic to that of S_1 to be lattice-isomorphic to S_1, it is necessary and sufficient that every direct factor of S_1 be self-dual. Theorem 1 solves Problem 8 of G. Birkhoff [Lattice theory, Amer. Math. Soc. Colloq. Publ. v. 25, rev. ed., New York, 1948; MR **10**, 673] for finite or discrete modular lattices. The solution is based upon the fact that within a class of projective elements, a given graphical isomorphism either always preserves or always reverses inclusion. If all prime quotients whose order is preserved (or reversed, respectively) are annulled, a determining partition is obtained; the lattice of its classes is the A_1 (A_2) of condition (D). *P. M. Whitman* (Silver Spring, Md.).

Kontorovič, P. G., and Plotkin, B. I. Lattices with an additive basis. Mat. Sb. N.S. **35**(77), 187–192 (1954). (Russian)

Let S be a complete lattice and U a set of elements of S such that each element of S is the join of elements of U; then U is called an additive basis of S. An element $a\in S$ is called isolated in S (with respect to U) if for each $u\in U$, $a\cap u=u$ or $a\cap u=0$; the behavior of such elements is studied. If S has no isolated elements, it is called dense; a sufficient condition is that any two elements of U are contained in a third. The meet of all isolated elements containing a given $a\in S$ is called its isolator $I(a)$. For $u_0\in U$, $I(u_0)=\sum u_i$, summed over $u_i\in U$, $u_i\geqq u_0$, if (L_1) and (L_2) hold. (L_1) each $u\in U$ which is contained in the join of some set $V\subset U$ is contained in a finite join of elements of V; (L_2) if $u_1, u_2\in U$, $u_1\cap u_2\neq 0$, then, for some $u_3\in U$, $u_1\leqq u_3$, $u_2\leqq u_3$. $V\subset S$ is called a covering of S if for each $u\in U$, there exists $v\in V$ with $u\leqq v$; a covering is called a decomposition basis if the intersection of pairs of distinct elements is always 0. If (L_1) and (L_2) hold, then S has a decomposition basis. If S has a decomposition basis, it has an irreducible such basis. The relation of modular elements to these concepts is investigated. Finally, this work is applied to nilpotent (and locally nilpotent) groups without torsion; a lattice-theoretic characterization of such groups is obtained. If two groups are lattice-isomorphic and one is locally nilpotent without torsion, so is the other, and the properties of certain corresponding subgroups can be correlated. [See Plotkin, Mat. Sb. N.S. **30**(72), 197–212 (1952); MR **13**, 908.] *P. M. Whitman* (Silver Spring, Md.).

Schützenberger, Marcel Paul. Un treillis universel des géométries projectives. C. R. Acad. Sci. Paris **239**, 1754–1756 (1954).

A modular lattice L is called a universal lattice for projective geometries if for each simple field K there is a lattice-homomorphism of L onto the lattice of the linear varieties of the projective space of dimension n over K. Necessary and sufficient conditions that a given set of $n+2$ elements generate a universal lattice for projective geometries with no non-trivial distributive homomorphic image are found in terms of the equality and inequality of certain complicated lattice-polynomials. *P. M. Whitman.*

Mal'cev, A. I. On the general theory of algebraic systems. Mat. Sb. N.S. **35**(77), 3–20 (1954). (Russian)

An algebraic system is a set with given finitary operations. By iterating these operations we get the polynomials of the system, which may be regarded as functions of some of the variables, the others being held fixed. Polynomials in one variable are translations. A primitive class is the set of all algebras defined by certain operations and relations. Typical of the results in §1 is the following: in order that all congruence relations on the algebras of a primitive class commute it is necessary and sufficient that there exist a polynomial $f(x, y, z)$ satisfying $f(x, x, y)=y$, $f(x, y, y)=x$. To prove the necessity one works in the free algebra generated by x, y and z. Let θ, ϕ be the congruence relations resulting from setting $x=y$, $y=z$. Then the commutativity of θ and ϕ gives us w with $x\equiv w(\phi)$, $w\equiv y(\theta)$; take $f(x, y, z)=w$. A normal complex is an equivalence class under a congruence. §2 is concerned with conditions for a collection of sets to be normal complexes and conditions under which a single normal complex determines the congruence uniquely; these conditions are stated in terms of the effect of translations.

The final three sections of the paper are devoted to topological algebraic systems. Here one postulates an algebra, a topological space, and joint continuity of the operations. It is shown that any topological loop is regular and that a connected loop is generated by any neighborhood of the identity. It is pointed out that these results extend to any algebra in which there exist binary polynomials giving the

multiplication and division of a loop, and this observation is phrased in terms of primitive classes. If an algebra A has a one-element subalgebra e and is generated by a connected neighborhood of e, then A itself is connected. A satisfactory theory of reduction of a topological algebra modulo a congruence is possible if we have the ternary operation associated with commuting congruence relations. The theorem of Pontrjagin, asserting the bicontinuity of a continuous isomorphism between locally compact groups satisfying the second axiom of countability, is extended to algebras with two ternary operations xyz and $xTyTz$ satisfying $(xyz)TyTz=x$, $(xTyTz)yz=x$, $xxz=z$. The existence of a covering algebra is proved for arcwise connected locally simply connected algebras with a one-element subalgebra. It is shown that every discrete normal subloop of a connected topological loop is central, and the final theorem is the commutativity of the fundamental group, proved under the assumption of a binary operation with an identity element. *I. Kaplansky* (Los Angeles, Calif.).

Szász, Gábor. The question of independence of the associativity conditions in the commutative multiplicative case. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3, 97–109 (1954). (Hungarian)

Hungarian version of Acta Sci. Math. Szeged 15, 130–142 (1954); MR 15, 773.

Fuchs, László. On the principal theorem of ideal theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3, 87–95 (1954). (Hungarian)

Hungarian version of a paper in Acta Math. Acad. Sci. Hungar. 5, 95–99 (1954); MR 16, 5.

Steinfeld, Ottó. Remark on a paper of N. H. McCoy. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3, 145–147 (1954). (Hungarian)

Hungarian version of Publ. Math. Debrecen 3, 171–173 (1954); MR 15, 773.

Andrunakievič, V. A. Rings with minimality condition for ideals. Dokl. Akad. Nauk SSSR (N.S.) 98, 329–332 (1954). (Russian)

After M. Hall [Trans. Amer. Math. Soc. 48, 391–404 (1940); MR 2, 122] we say that a ring is bound to its Brown-McCoy radical N if the two-sided annihilator N^* of N is contained in N. Let A be any ring with the descending chain condition on two-sided ideals. The author proves that A can be written as $B\oplus C$ with B a direct sum of a finite number of simple rings with unit, and C bound to its radical. The crucial part of the argument is to prove that $N\cap(N^*)^2$ is 0. A similar argument was given by Brown and McCoy under the hypothesis that A/N is regular [Proc. Amer. Math. Soc. 1, 165–171 (1950); MR 11, 638]. The reviewer remarks that the following hypothesis is adequate and covers both cases: assume that in A/N the two-sided ideal generated by any element a contains a left unit for a. *I. Kaplansky.*

Ukegawa, Takasaburo. Über zum Führer teilerfremde Ideale einer Ordnung. J. Inst. Polytech. Osaka City Univ. Ser. A. 5, 71–73 (1954).

Let S be a ring with unity element, and $\mathfrak{o}$ and $\mathfrak{o}^*$ be equivalent orders of S with $\mathfrak{o}\subseteq\mathfrak{o}^*$. Let G^* (G) be the semigroup of all $\mathfrak{o}^*$-ideals ($\mathfrak{o}$-ideals) relatively prime to $\mathfrak{F}$, where $\mathfrak{F}$ is the largest $\mathfrak{o}^*$-ideal such that $\mathfrak{o}^*\mathfrak{F}\subseteq\mathfrak{o}$ and $\mathfrak{F}\mathfrak{o}^*\subseteq\mathfrak{o}$. The author proves that G is a group if G^* is, and other related results. *R. E. Johnson* (Northampton, Mass.).

Fuller, L. E. A canonical set for matrices over a principal ideal ring modulo m. Canad. J. Math. 7, 54–59 (1955).

Let P be a principal-ideal ring, p be a prime in P, and R be the ring $P/(p^k)$. This paper gives a unique canonical form for the matrices of R_n under left multiplication by unimodular matrices. This canonical form is a generalization of the Hermite form for matrices over a field. *R. E. Johnson* (Northampton, Mass.).

Seidenberg, A. On the dimension theory of rings. II. Pacific J. Math. 4, 603–614 (1954).

[For part I see same J. 3, 505–512 (1953); MR 14, 941.] An integral domain O is said to be of dimension n if in O there is a properly ascending chain of prime ideals $(0)\subset P_1\subset P_2\cdots\subset P_n\subset(1)$ but no chain of the form $(0)\subset P_1'\subset P_2'\cdots\subset P'_{n+1}\subset(1)$. In a previous paper the author showed that if O is n-dimensional, then the dimension m, of $O[x]$, where x is transcendental over O, must satisfy $n+1\leqq m\leqq 2n+1$. In this paper the author proves that for every m satisfying $n+1\leqq m\leqq 2n+1$ one can find an integral domain O of dimension n so that $O[x]$ has dimension m. If one further restricts O to be a multiplication ring (that is, one for which O_p is a valuation ring for each prime ideal P in O), then if the dimension of O is m, the dimension of $O[x_1,\cdots,x_n]$ is exactly $m+n$. An F-ring is an integral domain O of dimension 1 for which $O[x]$ has dimension 3. For F-rings O the author proves that $O[x_1,\cdots,x_n]$ has dimension m with $n+2\leqq m\leqq 2n+1$, and all such values can be taken on. The author also relates the dimension in various situations with the degree of transcendence. *I. N. Herstein* (Philadelphia, Pa.).

Shimura, Goro. A note on the normalization-theorem of an integral domain. Sci. Papers Coll. Gen. Ed. Univ. Tokyo 4, 1–8 (1954).

By the "normalization theorem" is meant the fact that, if $K[x_1,\cdots,x_n]$ is a finite integral domain of transcendence degree d, there exist d elements $y_1,\cdots,y_d$ in $K[x]$ such that $K[x]$ is integral over $K[y]$. The extension of this result to the case where K is replaced by an integral domain $\mathfrak{o}$ is studied in the general set-up of the reduction of algebraic varieties with respect to a discrete valuation, under the additional assumption that (x) is homogeneous over the quotient field K of $\mathfrak{o}$ (i.e. is the homogeneous generic point of some projective variety over K). Suppose further that $\mathfrak{o}$ is a discrete valuation ring. Then, in order for $\mathfrak{o}[x]$ to be integral over $\mathfrak{o}[y]$ ($(y_0,\cdots,y_r)$ being a set of homogeneous of like degree in the x_i's), it is necessary and sufficient that (0) be the only specialization of (x) extending $(y)\rightarrow(0)$ and the canonical homomorphism of $\mathfrak{o}$ onto its residue field k. After noticing that, if an intersection of projective varieties over K is empty, then the intersection of the corresponding "reduced" varieties over k is also empty, the existence of such a set $(y_0,\cdots,y_r)$ is easily proved.

Consider now the ring I of integers in some algebraic number field K. It is proved that, given a projective variety V^r over K, there exist $r+1$ hypersurfaces H_i such that, for every prime $\mathfrak{p}$ of K, the intersection of the $r+2$ varieties $V^{\mathfrak{p}}$, $H_i^{\mathfrak{p}}$ reduced mod $\mathfrak{p}$ is empty. This proves the normalization theorem for any homogeneous integral domain $I[x]$. The validity of this result depends essentially upon the fact that every ideal class in K is of finite order; counter-examples may be found if one takes for I a suitable ring of integral functions in one variable over an infinite field. *P. Samuel* (Cambridge, Mass.).

Hion, Ya. V. Archimedean ordered rings. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 237–242 (1954). (Russian)

Let A be a ring which, as an additive group, is simply ordered and Archimedean. Assume that $a \geqq 0$ and $b \geqq 0$ imply $ab \geqq 0$. Then either A is isomorphic to a ring of real numbers, or else A is isomorphic to an additive group of real numbers with all products 0. Moreover, in the first case the embedding is unique and in the second it is unique up to multiplication by a positive real number. The proof begins with the knowledge that A is additively isomorphic to a group of real numbers, and then proceeds by using the fact that any order-isomorphism is given by multiplication by a positive real number. *I. Kaplansky* (Los Angeles, Calif.).

Kokoris, Louis A. New results on power-associative algebras. Trans. Amer. Math. Soc. **77**, 363–373 (1954).

Albert's general structure theory for commutative power-associative algebras [same Trans. **69**, 503–527 (1950); MR **12**, 475] carries the restriction that the characteristic of the base field be $\neq 2, 3, 5$. In this paper, by strengthening the assumption of power-associativity to that of strict power-associativity (A is strictly power-associative in case A_K is power-associative for every scalar extension K of the base field), the author proves that Albert's theorems hold for fields of characteristic $\neq 2$. The crucial results are: (1) Every simple commutative strictly power-associative algebra of characteristic $\neq 2$ is either an algebra of degree one or two with a unity quantity or is a Jordan algebra. (2) If e is a principal idempotent in a commutative strictly power-associative algebra A of characteristic $\neq 2$, then $A_e(\frac{1}{2})+A_e(0)$ is contained in the radical of A. The other results in the general theory follow by Albert's proofs from (1) and (2). A definition of strict power-associativity for rings is given, together with proofs, for rings of characteristic prime to 2, of analogues of Albert's theorems on power-associative rings of characteristic prime to 30. The paper concludes with a theorem on power-associative algebras of degree 2 and characteristic 0 which the author hopes will be useful in proving the conjecture that all simple commutative power-associative algebras of degree 2 and characteristic 0 are Jordan algebras. *R. D. Schafer* (Storrs, Conn.).

Eilenberg, Samuel. Algebras of cohomologically finite dimension. Comment. Math. Helv. **28**, 310–319 (1954).

Results of Ikeda, Nagao and Nakayama concerning algebras of finite cohomology dimension [Nagoya Math. J. **7**, 115–131 (1954); MR **16**, 214] are reformulated and reproved in the terms of general "homological algebra" as developed by H. Cartan and the author [Homological algebra, Princeton Univ. Press, 1955 (in press)].

Let R be a ring with an identity element which is to act as the identity transformation on all R-modules considered. The projective dimension $p_R(A)$ of an R-module A is defined as the least non-negative integer n (or as ∞, if no such integer exists) for which there exists an exact sequence $(0)\rightarrow X_n \rightarrow \cdots \rightarrow X_0 \rightarrow A \rightarrow (0)$ in which all the R-modules X_i are R-projective. The global dimension $d(R)$ of the ring R is defined as sup $(p_R(A))$, as A ranges over all R-modules.

If L is an algebra (with identity element) over a field K, the cohomology dimension $c(L)$ of L is the largest non-negative integer n (or ∞, if no such integer exists) for which there exists a two-sided L-module A such that the n-dimensional cohomology group for L in A is not (0). If L^* denotes the algebra anti-isomorphic with L, a two-sided L-module is nothing but an ordinary (left) module for $L\otimes L^*$, the tensor product being taken with respect to K. The n-dimensional cohomology group for L in A has been identified (in the book of H. Cartan and the author referred to above) with the group $\mathrm{Ext}^n{}_{L\otimes L^*}(L, A)$, where "Ext" is one of the basic functors with which homological algebra is concerned. It follows very easily from this that $c(L) = p_{L\otimes L^*}(L)$.

Assume now that L is of finite dimension over K, and let P denote the factor algebra of L with respect to its radical. The main results of the present paper are then the following three theorems: (1) $c(L) = d(L\otimes P^*) = p_{L\otimes P^*}(P)$. (2) If P is separable then $c(L) = p_L(P)$. (3) If $c(L)$ is finite then P is separable.

While theorems (1) and (2) follow from general results of homological algebra, the proof of theorem (3) uses in addition specific algebraic theory; in particular, the Cartan invariants of an algebra which were used also by Ikeda, Nagao and Nakayama [loc. cit.]. The results of these last authors include (3), while those corresponding to (1) and (2) are less conceptual and somewhat less incisive, in that they refer to a series of certain special L-modules (constructed explicitly from L) rather than to the more obviously significant global and projective dimensions of $L\otimes P^*$ and P alone. *G. Hochschild* (Urbana, Ill.).

Nagao, Hirosi. On $\mathfrak{l}$-relative cohomology groups of an associative algebra. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 15–29 (1954).

The relative cohomology groups of an algebra with respect to a left ideal, as introduced by Nakayama [Nagoya Math. J. **6**, 177–185 (1953); MR **15**, 393] and used by the present author, Ikeda and Nakayama in their joint work on algebras with vanishing n-cohomology groups [ibid. **7**, 115–131 (1954); MR **16**, 214], are discussed in greater detail here. The earlier results are then extended to similar results characterizing algebras with vanishing relative n-cohomology groups. *G. Hochschild* (Urbana, Ill.).

Adamson, Iain T. Cohomology theory for non-normal subgroups and non-normal fields. Proc. Glasgow Math. Assoc. **2**, 66–76 (1954).

Let G be a group, and let K be a subgroup of finite index in G. For any G-module A, the author defines the relative cohomology groups for $G \bmod K$ in A, denoted $H^n([G:K], A)$, using a basic acyclic G-complex which is obtained from the usual homogeneous G-complex by replacing m-tuples of elements of G with m-tuples of cosets σK ($\sigma \in G$). If K is normal in G one has

$$H^n([G:K], A) = H^n(G/K, A^K),$$

where A^K is the K-fixed part of A, regarded as a G/K-module. A further result which is important for the author's purpose is that the usual exact cohomology sequence holds for the relative groups with respect to all G-module extensions $(0)\rightarrow A' \rightarrow A \rightarrow A'' \rightarrow (0)$ whose kernels A' satisfy $H^1(U, A') = (0)$ for all subgroups U of K. In the situation of class-field theory, where the groups are Galois groups and the modules are the multiplicative groups of fields, ideles or idele classes, the above results, together with the existence of suitable splitting modules for the relative cohomology groups, are used to extend the definitions of the cohomology groups figuring in the class-field theory from the case of Galois extensions to arbitrary finite separable algebraic extensions, and to reduce the determination of these new groups to that of the usual ones. If E/k is a finite separable algebraic extension of k, and if U/k is (for instance) the least Galois extension containing it (actually, the choice of U is shown to be irrelevant), then the groups for E/k are

defined as the relative groups for the Galois group of U/k mod the subgroup corresponding (in the Galois correspondence) to the intermediate field E. *G. Hochschild.*

Hochschild, G. Cohomology classes of finite type and finite dimensional kernels for Lie algebras. Amer. J. Math. **76**, 763–778 (1954).

Let L be a Lie algebra and M a finite-dimensional module over L. A cohomology class c in $H(L, M)$ has been called effaceable by Koszul if there exists a finite-dimensional L-module N containing M such that the image of c under the injection homomorphism $H(L, M)\rightarrow H(L, N)$ is 0. Koszul has proved that, if L is solvable (of characteristic 0) then, for any M, every element of dimension >0 of $H(L, M)$ is effaceable [same J. **76**, 535–554 (1954); MR **15**, 928]. In the present paper this theorem is proved by another method and is generalized as follows. Let K be any Lie algebra over a field of characteristic 0, L its radical and $K=S+L$ a Levi decomposition of K. Let M be a finite-dimensional K-module; then, a cohomology class c in $H(K, M)$ is effaceable if and only if its restriction to S is 0.

Let K be a Lie algebra and M a finite-dimensional K-module. We may consider M as a module over the enveloping algebra R_K of K, and $H(K, M)$ is then the cohomology space $H(I, M)$, where I is the ideal generated by K in R_K. An element c of $H(K, M)$ is called of finite type if it belongs to the canonical image in $H(I, M)$ of $H(I/C, M)$, C being some ideal of finite codimension of R_K contained in I and annihilating M. Every cohomology class of finite type and of dimension >0 is effaceable, but it is shown by an example that the converse is not true. However, every cohomology class of dimension 0, 1 or 2 is of finite type, and a cohomology class of dimension 3 which is effaceable is of finite type. If K is solvable (over a field of characteristic 0), then every cohomology class of dimension >0 of K is of finite type.

These results are then applied to the determination of the finite-dimensional "kernels" [for the definitions relative to kernels, cf. Hochschild, ibid. **76**, 698–716 (1954); MR **16**, 109]. Let K be a Lie algebra over the field F and N a finite-dimensional K-module; for a cohomology class c in $H^3(K, N)$ to be the obstruction of a finite-dimensional kernel with center N, it is sufficient in general and also necessary if F is of characteristic 0 that c be effaceable.

C. Chevalley (New York, N. Y.).

Andrunakievič, V. A. The radical in generalized Q-rings. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 419–426 (1954). (Russian)

If we assume a unit element, a generalized Q-ring is a topological ring A in which there exists a neighborhood of 1 consisting of elements x satisfying $AxA=A$. This coincides with Iseki's G-rings [An. Acad. Brasil. Ci. **25**, 79–86 (1953); MR **14**, 720], and the two papers contain very similar results. The author observes in a footnote that his results were obtained before seeing Iseki's announcement [C. R. Acad. Sci. Paris **234**, 1938–1939 (1952); MR **13**, 815].

I. Kaplansky (Los Angeles, Calif.).

Theory of Groups

Thurston, H. A. Some properties of partly-associative operations. Proc. Amer. Math. Soc. **5**, 487–497 (1954).

Etude de certaines relations entre diverses extensions aux operations algébriques n-aires des concepts d'associativité, de régularité et de réversibilité basée sur un travail antérieur de l'auteur [J. London Math. Soc. **24**, 260–271 (1949); MR **11**, 316] et utilisant certaines analogies avec les notions et propriétés introduites par E. L. Post dans sa théorie des groupes polyadiques [Trans. Amer. Math. Soc. **48**, 208–350 (1940); MR **2**, 128]. *J. Riguet* (Paris).

Thierrin, Gabriel. Sur la caractérisation des groupes par certaines propriétés de leurs relations d'ordre. C. R. Acad. Sci. Paris **239**, 1453–1455 (1954).

Let S be a semigroup with cancellation on both sides. The author proves that S is a group if and only if its right regular order relations are cancellable on the right; likewise if and only if its right cancellable order relations are right regular. The term "order relation" is not defined, but the proofs indicate that the author means thereby any reflexive and transitive relation. A relation R is called right regular if aRb implies $acRbc$, and right cancellable if $acRbc$ implies aRb. *A. H. Clifford* (Baltimore, Md.).

Thierrin, Gabriel. Sur quelques propriétés de certaines classes de demi-groupes. C. R. Acad. Sci. Paris **239**, 1335–1337 (1954).

A homogroup is a semigroup (French: "demi-groupe") containing an ideal which is also a subgroup. A semigroup D is called reversible if, for every a and b in D, neither $aD\cap bD$ nor $Da\cap Db$ is empty. D is called strongly reversible if, for every a and b in D, there exist positive integers r, s, t such that $(ab)^r=a^sb^t=b^ta^s$. The following theorems are proved. A semigroup without zero is a homogroup if and only if it is reversible and contains a minimal right ideal and a minimal left ideal. Every strongly reversible semigroup, and also every finite semigroup, is the union of disjoint subsemigroups no one of which contains a proper two-sided prime ideal of itself. A commutative semigroup A is without proper prime ideals if and only if, for every a and b in A, a divides some power of b.

A. H. Clifford (Baltimore, Md.).

Kimura, Naoki. Maximal subgroups of a semigroup. Kōdai Math. Sem. Rep. **1954**, 85–88 (1954).

The principal result of this paper is that each idempotent element of a semigroup S is contained in a unique maximal subgroup of S, and any two distinct maximal subgroups of S are disjoint. This was proved for torsion semigroups by Š. Schwarz [Sb. Prác Prirod. Fak. Slovensk. Univ. Bratislave no. **6** (1943); MR **10**, 12]. *A. H. Clifford.*

Kimura, Naoki. On some examples of semigroups. Kōdai Math. Sem. Rep. **1954**, 89–92 (1954).

Two examples are given of left simple semigroups obeying the left cancellation law. One of these is the same as that of R. Baer and F. Levi [S.-B. Heidelberger Akad. Wiss. **1932**, no. 2, 3–12, p. 7]. *A. H. Clifford* (Baltimore, Md.).

Tamura, Takayuki. Note on unipotent inversible semigroups. Kōdai Math. Sem. Rep. **1954**, 93–95 (1954).

A semigroup S is called "unipotent" if it contains exactly one idempotent e, and "left (right) inversible" if to each $a\in S$ there exists $x\in S$ such that $xa=e$ $(ax=e)$. If S is left inversible it is also right inversible, and vice-versa, and the Suschkewitsch kernel of S is the same as the maximal subgroup G of S containing e (group of zeroids of S). Let M be the Rees factor semigroup S/G; M is a "zero-semigroup" in the sense that it contains a zero element but no other idempotent. Using the reviewer's extension theory [Trans.

Amer. Math. Soc. **68**, 165–173 (1950); MR **11**, 499] the author points out that the structure of S is completely determined by M, G, and a ramified homomorphism f of M into G. If $S_i=(M_i, G_i, f_i)$ $(i=1, 2)$, a necessary and sufficient condition is given that S_1 and S_2 be isomorphic; this theorem applies also if S_1 and S_2 are any semigroups with zeroids. *A. H. Clifford* (Baltimore, Md.).

Sabidussi, Gert. Loewy-groupoids related to linear graphs. Amer. J. Math. **76**, 477–487 (1954).

Soit G un graphe considéré comme ensemble de points et de segments satisfaisant à une relation d'incidence. H étant un sous graphe de G, l'auteur désigne par $M(G, H)$ l'ensemble des applications de H dans G qui transforme les points en points, les segments en segments, et conserve la relation d'incidence, $M_0(G, H)$ (respectivement $M_1(G, H)$) étant l'ensemble des restrictions des applications de $M(G, H)$ à l'ensemble des points de H (respectivement à l'ensemble des segments de H). Il montre que $M(G, H)$, $M_0(G, H)$, $M_1(G, H)$ sont des groupoïdes à gauche au sens de Loewy [J. Reine Angew. Math. **157**, 239–254 (1927)] et discute les relations que ces groupoïdes présentent entre eux.

La seconde partie du travail est consacrée à un théorème de représentation de groupoïdes à gauche par des groupoïdes de type $M(G, H)$ dont l'énoncé exact est trop long pour être reproduit ici. *J. Riguet* (Paris).

Erdős, Jenő. The theory of groups of finite class. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 127–143 (1954). (Hungarian)

Hungarian version of Acta Math. Acad. Sci. Hungar. **5**, 45–58 (1954); MR **16**, 217.

Howson, A. G. On the intersection of finitely generated free groups. J. London Math. Soc. **29**, 428–434 (1954).

In this paper it is proved that the intersection $U\cap V$ of two subgroups U and V of a free group F having respectively m and n free generators is finitely generated with at most $2mn-m-n+1$ free generators. It can be assumed that F is a free group on two generators. The proof of the theorem is accomplished by exploiting the topological properties of the Dehn diagram of F. *D. G. Higman.*

Sesekin, N. F., and Starostin, A. I. On a class of periodic groups. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 225–228 (1954). (Russian)

A locally finite group, every Sylow p-group of which is locally cyclic, is called a C-group. If a group G is a semi-direct product of two locally cyclic periodic groups, A by B, the orders of whose elements are relatively prime, then G is a C-group and A contains the commutant of G. Conversely, a C-group G is decomposable as a semi-direct product of two locally cyclic subgroups, G' by B, such that G' is the commutant of G and the order of every element of G' is relatively prime to the order of every element of B. *R. A. Good* (College Park, Md.).

Plotkin, B. I. On some criteria of locally nilpotent groups. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 181–186 (1954). (Russian)

In a group an element g is called a nilelement provided to every element h there corresponds a natural number n such that the nth iterated commutator $[\cdots[[h, g], g]\cdots, g]=1$. A group is a nilgroup provided each of its elements is a nilelement. [Compare terminology of weakly central element [Shenkman, Pacific J. Math. **3**, 501–504 (1953); MR **15**, 9] and Engel condition [Plotkin, Dokl. Akad. Nauk SSSR (N.S.) **94**, 999–1001 (1954); MR **15**, 682].] Conditions under which a nilgroup is locally nilpotent are investigated. One such condition is that the nilgroup have an ascending solvable series. A group with category [Kontorovič, Mat. Sb. N.S. **28**(70), 79–88 (1951); MR **12**, 671] is a nilgroup and is locally nilpotent. A nilgroup, every abelian subgroup of which is finite, is itself finite and nilpotent. A torsion-free nilgroup, every abelian subgroup of which has finite rank, is nilpotent and has finite special rank [in the sense of Mal'cev, ibid. **22**(64), 351–352 (1948); MR **9**, 493]. *R. A. Good* (College Park, Md.).

Plotkin, B. I. On the nil-radical of a group. Dokl. Akad. Nauk SSSR (N.S.) **98**, 341–343 (1954). (Russian)

In a group $\mathfrak{G}$ the maximal locally nilpotent normal subgroup $R(\mathfrak{G})$ is called the nilradical of $\mathfrak{G}$. A series of subgroups $E=N_0\subset N_1\subset\cdots\subset N_\alpha\subset N_{\alpha+1}\subset\cdots\subset N_\gamma=\mathfrak{G}$ is called a nilseries for $\mathfrak{G}$ provided each N_α is invariant in $\mathfrak{G}$ and each factor of the series is locally nilpotent; if furthermore $N_{\alpha+1}/N_\alpha$ is, for every α, the nilradical of $\mathfrak{G}/N_\alpha$, the series is called the nilradical series. Each of the following three conditions is sufficient that the nilradical in a group $\mathfrak{G}$ coincide with the set of all nilelements [see the paper reviewed above] in $\mathfrak{G}$: (1) $\mathfrak{G}$ is locally solvable; (2) $\mathfrak{G}$ has an ascending normal solvable series; (3) $\mathfrak{G}$ has a nilradical series. *R. A. Good* (College Park, Md.).

★Kaplansky, Irving. Infinite abelian groups. University of Michigan Press, Ann Arbor, 1954. v+91 pp. $2.00.

In the last three decades the theory of infinite abelian groups experienced a rapid and extensive development to which we owe many fundamental results. In the course of this development new methods took shape which were soon of capital importance also in other modern branches of mathematics. Despite this, the vast literature of the subject could so far exhibit no monograph devoted to a systematic presentation of the results on infinite abelian groups scattered throughout the individual papers. The only work which brought a certain remedy to this shortcoming was the "Theory of groups" by Kuroš, the second edition of which [Gostehizdat, Moscow, 1953; MR **15**, 501] contains an extensive chapter of rich contents on infinite abelian groups. Now we have at hand the first genuine monographical presentation of this subject.

The task of writing such a systematic account was doubtless an arduous one, but Kaplansky has carried it out in a masterly way. He points out in the Introduction that his monograph has been written with two objectives in mind: first, to make the theory of infinite abelian groups available in a convenient form to the mathematical public; second, to help students acquire some of the techniques used in modern infinite algebra. For this latter purpose the study of infinite abelian groups is particularly suitable since no extensive background is required, the rudiments of group theory being sufficient. At the same time there are many possibilities for the use of transfinite methods since the theory of infinite abelian groups abounds in typical applications of Zorn's lemma.

The first thing which impresses the reader is the condensation of a wealth of material into the relatively small extent of the book without detriment to the completeness and clearness of the presentation. The discussion is occasionally somewhat concise; a large part of the material is contained in consecutive examples, and yet the author always succeeds

in saying just as much as is needed. Thus his work serves both as an excellent introduction for the beginner and as a complete account of the subject for the advanced algebraist. Especial mention must be made of the author's ability to point out what is essential in the material treated and to clear up its significance.

The following example is characteristic of the author's didactical sense in introducing abstract concepts which at first would seem somewhat artificial. By his exposition the surprising content of Ulm's fundamental structure theorem becomes in a certain sense more tangible and natural. How can it be shown that an additive abelian group G which is a direct sum of cyclic p-groups admits essentially a unique representation as such a direct sum? Let P be the subgroup of G consisting of all elements $x \in G$ satisfying $px = 0$, and set $P_\alpha = P \cap p^\alpha G$ $(\alpha = 0, 1, 2, \cdots)$. Then the factor group $P_\alpha / P_{\alpha+1}$, as an elementary p-group, splits into a direct sum of a well defined cardinal number, say $f(\alpha)$, of groups of order p. One sees immediately that $f(\alpha)$ gives precisely the number of cyclic summands of order $p^{\alpha+1}$ in any decomposition of G into the direct sum of cyclic groups, proving so the invariance of this cardinal number. Now, on the other hand, Ulm's theorem states that the (suitably continued) transfinite sequence of the "Ulm invariants"

$$f(\alpha) \quad (\alpha = 0, 1, 2, \cdots, \omega, \omega+1, \cdots)$$

uniquely determines the structure of G also in the case when G is an arbitrary countable abelian p-group containing no subgroup of type p^∞. It is remarkable that this advantageous formulation of Ulm's theorem due to Mackey and Kaplansky operates with the Ulm invariants each of which is given by a single cardinal, while in the original formulation each of the "Ulm factors" is described by an infinity of cardinals.

After preliminaries about direct sums, important types of abelian groups (cyclic groups, rational group, the group of type p^∞), torsion groups and an exposition of Zorn's lemma, the book begins with an exhaustive study of divisible groups. It is shown that a divisible group is a direct summand of any containing group and that every divisible group is a direct sum of rational groups and of groups of type p^∞. That any group can be embedded in a divisible group the reader can prove for himself by solving an example for which the ground has been well prepared. Interrupting the exposition at this point, the author raises the question: how do we know when we have a satisfactory structure theorem for certain classes of groups? He suggests as a criterion of both theoretical and practical value the success of the alleged structure theorem in solving some deep explicit problems. Three such "test problems" are mentioned, one of which queries: Are G and H isomorphic groups if the direct sums $G+G$ and $H+H$ are isomorphic? (For example, Ulm's theorem proves satisfactory by this criterion since it answers in the affirmative the test problem mentioned for countable torsion groups.)

Next follows the exposition of Prüfer's theory of pure subgroups ("Servanzuntergruppe") presented in the elegant modern treatment of Kulikov. After proving the decomposability theorems on groups of bounded order, the author introduces the fundamental concept of height and deduces the basic facts about it. As immediate applications he obtains the important theorems of Kulikov: A mixed group cannot be indecomposable; an indecomposable torsion group is a cyclic p-group or a group of type p^∞. Then follows the theory of direct sums of cyclic groups, in particular Prüfer's fundamental theorem on the decomposability of countable p-groups without elements of infinite height into a direct sum of cyclic groups, and two fundamental theorems of Kulikov: A necessary and sufficient condition for an arbitrary p-group to be a direct sum of cyclic groups is that the subgroup of all elements of order p be the union of an ascending chain of subgroups of bounded height; any subgroup of a direct sum of cyclic groups is itself a direct sum of cyclic groups. The next section deals with Ulm's structure theory. Zippin's existence theorem is treated by a very clear and elegant method of Baer.

The following section extends all the preceding results to modules over principal-ideal rings and gives applications to the theory of linear transformations. Also, a number of results are obtained concerning linear transformations on infinite-dimensional vector spaces. A brief excursion is devoted to explain why there is virtually no connection between this algebraic theory of linear transformations and the theory of continuous linear transformations on Banach spaces.

The following three sections present the theory of torsion-free modules. The author proves all theorems with the greatest possible generality. He treats also a number of new results due to himself or his students. First, a very short and elegant proof is given for the fundamental theorem on finitely generated modules over a principal-ideal ring. Then it is shown that the additive group of p-adic integers is indecomposable. The existence of indecomposable torsion-free R-modules of rank two is proved for an arbitrary principal-ideal ring R which is not a complete discrete-valuation ring. [A principal-ideal ring R is called a discrete-valuation ring if it contains exactly one prime element p (up to unit factors). R is said to be complete if it is complete in the topology defined by the system of ideals (p^n) as a neighborhood basis of 0.] If R is a complete discrete-valuation ring, then any countably generated torsion-free R-module is a direct sum of a divisible module and of a free module. Here the countability assumption cannot be omitted. Namely, if R is a principal-ideal ring, but not a field, then the complete direct sum of a countable number of copies of R is no free module. If R is a complete discrete-valuation ring and M an R-module without elements of infinite height which is complete in its p-adic topology, then M is the completion of a direct sum of cyclic modules. For such a ring R any reduced R-module has a cyclic direct summand, and any indecomposable R-module is isomorphic to one of the following modules: R, $R/(p^n)$, K, K/R, where K denotes the quotient field of R. As an application the author determines all abelian groups which can be equipped with a compact topology.

The next two sections contain the author's new results on characteristic and fully invariant submodules as well as on the endomorphism ring $E(M)$ of a primary R-module M over a complete discrete-valuation ring R. The submodules mentioned are characterized in terms of the "Ulm sequence" of an element. If M and N are faithful primary R-modules, then any R-isomorphism between $E(M)$ and $E(N)$ is induced by an isomorphism of M and N. The center of $E(M)$ is precisely R. Any automorphism of $E(M)$ which leaves the center fixed elementwise is inner. Also, an application is given to linear transformations. The book concludes with a valuable guide to the literature and a bibliography containing 9 books and 145 papers.

This book constitutes a remarkable enrichment of the literature on modern algebra. It gives the most important results on the subject presented from the most modern point

of view. A great part of the material treated consists of entirely new, hitherto unpublished, results of the author and his students. The only criticism the reviewer can mention is that no treatment is given of Kulikov's basic subgroup which has proved to be a most powerful tool in investigating abelian p-groups. This excellent book will certainly initiate a new development of the theory of abelian groups.

T. Szele (Debrecen).

Asano, Keizo. Bemerkungen über die Erweiterungstheorie von Gruppen. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 75–80 (1954).

Takahasi, Mutuo. Group extensions and their splitting groups. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 81–85 (1954).

A group G is termed an extension of N by Γ, if N is a normal subgroup of G and $G/N \simeq \Gamma$. Groups containing all extensions of G by Γ may be constructed in many ways; one may form their direct product or their free product or their free product with amalgamated N, etc. In these two papers another construction is proposed: Denote by $H=H(N, \Gamma)$ the group of all single-valued mappings of Γ into the holomorph of N. If h is an element in H, and if σ, ν are elements in Γ, then let $h(\nu)^\sigma = h(\nu\sigma^{-1})$; and in this way the elements in Γ induce automorphisms in H. The group $U=U(N, \Gamma)$ is then the splitting extension of H by Γ which realizes these induced automorphisms. This group U may be realized in an obvious fashion as a group of permutations of the pairs (n, σ) for n in N and σ in Γ; and it contains all extensions of N by Γ (up to isomorphisms). If N^* is the subgroup of all single-valued mappings of Γ into N, then N^* is a normal subgroup of H and U. If the extension G of N by Γ is contained in U, then N^*G is a splitting extension of N^* by Γ.

R. Baer (Urbana, Ill.).

Gaschütz, Wolfgang. Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden. Math. Z. **60**, 274–286 (1954).

Let $\mathfrak{g}$ be a group of finite order N. If $\mathfrak{g}$ can be generated by e elements, then $\mathfrak{g}$ is a homomorphic image of a free group $\mathfrak{F}$ with e free generators, say $\mathfrak{g} \cong \mathfrak{F}/\mathfrak{f}$ where the normal subgroup $\mathfrak{f}$ of $\mathfrak{F}$ is a free group with $e'=1+(e-1)N$ free generators [cf. O. Schreier, Abh. Math. Sem. Hamburg. Univ. **5**, 161–183 (1927)]. Let $\mathfrak{f}_p$ be the least normal subgroup of $\mathfrak{f}$ for which $\mathfrak{f}/\mathfrak{f}_p$ is abelian of prime exponent p. Then $\mathfrak{f}_p$ is normal in $\mathfrak{F}$. Every inner automorphism of $\mathfrak{F}$ defines an automorphism of $\mathfrak{f}/\mathfrak{f}_p$ and hence a linear transformation of an e'-dimensional vector space over the field $\Gamma^{(p)}$ with p elements. In this manner, a representation $F_e^{(p)}$ of $\mathfrak{g}$ of degree e' over the field $\Gamma^{(p)}$ is obtained. It is shown that this representation depends only on $\mathfrak{g}$, e, and p, but not on the particular choice of the homomorphism of $\mathfrak{F}$ on $\mathfrak{g}$. The irreducible constituents of $F_e^{(p)}$ are the same which appear in the $(e-1)$-times repeated regular representation R of $\mathfrak{g}$ over the field $\Gamma^{(p)}$ plus an additional 1-representation. If p does not divide N, then because of the complete reducibility of the representations of $\mathfrak{g}$ over the field $\Gamma^{(p)}$, the representation $F_e^{(p)}$ is completely determined by this result. For an arbitrary choice of p, it is shown that $F_{e+1}^{(p)}$ is the direct sum of $F_e^{(p)}$ and R. Let $U_1, U_2, \cdots, U_t$ denote the non-similar indecomposable representations which can be obtained by direct decomposition of R. The author obtains a direct decomposition of $F_e^{(p)}$ into components U_i and a further component H which does not possess a component U_i. The multiplicity of each U_i in $F_e^{(p)}$ is given explicitly. While certain information is available concerning H, the complete structure of H is not known. The decomposition of the representation module of $F_e^{(p)}$ into a direct sum of the representation module of H and another module can be characterized in terms of certain extensions of abelian groups of exponent p by means of $\mathfrak{g}$. Some applications are given concerning the number of generators of finite groups. For instance, necessary and sufficient conditions are obtained that an extension of a finite p-group by means of $\mathfrak{g}$ can be generated by e elements.

R. Brauer.

Bauer, Friedrich L. Zur Theorie der Spingruppen. Math. Ann. **128**, 228–256 (1954).

The relation between the representation theory of the classical groups and that of the symmetric group has long been known. In the case of the orthogonal group ${}^nO$ of n dimensions an irreducible representation is denoted by ${}^nO(m_i)$, where $m_i \geqq m_{i+1}$ $(i=1, 2, \cdots)$ with $n=2\nu$ or $2\nu+1$. Moreover, the m_i are all integers or every m_i is half an odd integer. The Young diagram $[m_i]$ representing an 'integral' representation is of the familiar type while in the 'half-integral' case the diagram contains a bottom row of $\frac{1}{2}$ squares (contrary to usual custom, the author identifies a Young diagram by its columns rather than its rows, effectively rotating it through 90° in a counter clockwise direction. This has the virtue that the identity representation of S_n is associated with the identity class of the group). The particular half-integral representation for which each $m_i=\frac{1}{2}$, called the Clifford representation, has degree 2^ν and is denoted ${}^nO(\Delta)$. The Kronecker product (direct product) of N such representations is denoted by ${}^nO(\Delta)_\times{}^N$ and the reduction of this representation is the problem under consideration.

To obtain this reduction the author defines the N-complement [cf. Bauer, C. R. Acad. Sci. Paris **234**, 1743–1744 (1952); MR **14**, 16] of ${}^nO(m)$ as the representation

$$ {}^NO\left(\left(\frac{n}{2}\right)^{\alpha_0}, \left(\frac{n-2}{2}\right)^{\alpha_1}, \left(\frac{n-4}{2}\right)^{\alpha_2}, \cdots, 0^{\alpha_\nu}\right) \quad \text{for} \quad n=2\nu, $$

$$ {}^NO\left(\left(\frac{n}{2}\right)^{\alpha_0}, \left(\frac{n-1}{2}\right)^{\alpha_1}, \left(\frac{n-3}{2}\right)^{\alpha_2}, \cdots, \left(\frac{1}{2}\right)^{\alpha_\nu}\right) \quad \text{for} \quad n=2\nu+1, $$

where $\alpha_0=\frac{1}{2}N-m_1$, $\alpha_i=m_i-m_{i+1}$, $\alpha_\nu=m_\nu$ (N even), $m_\nu-\frac{1}{2}$ (N odd). (There are some unfortunate misprints in these formulae at the beginning of §3 against which the reader should be warned.) After illustrating the process of N-complementation the author states his structure theorem (I): In the reduction of ${}^nO(\Delta)_\times{}^N$ the irreducible representation ${}^nO(m)$ appears with a multiplicity equal to the degree $(n=2\nu)$, or the reduced degree $(n=2\nu+1)$, of its N-complement. Since these degrees are known this theorem tells the whole story and is to be compared with Schur's celebrated result for the full linear group. The proof is by induction, utilizing the convenient shift in emphasis between n and N.

The remainder of the paper is devoted to the implications of the theorem for the commuting algebra of Brauer and Weyl [Amer. J. Math. **57**, 425–449 (1935)] and for invariant theory. Theorem II is as follows: The commuting algebra of ${}^{2\nu}O(\Delta)_\times{}^N$ is isomorphic to the enveloping algebra of ${}^NO^+(\Delta)_\times{}^{2\nu}$, and similarly for ${}^{2\nu}O^+(\Delta)_\times{}^N$ and ${}^NO(\Delta)_\times{}^{2\nu}$ (where the $+$ indicates the subgroup of rotations). If we set ${}^nO(\Delta)_\times{}^2={}^nO(\square)$, then a similar result (IIa) holds for ${}^{2\nu}O(\square)_\times{}^M$ and ${}^{2M}O^+(\square)_\times{}^\nu$ and for ${}^{2\nu}\mathrm{Sp}(\square)_\times{}^M$ and ${}^{2M}\mathrm{Sp}(\square)_\times{}^\nu$. Finally the author shows that his structure theorem also yields the reduction of ${}^n\mathrm{SL}(\square)_\times{}^M$.

G. de B. Robinson.

Yamanoshita, Tsuneyo. On the dimension of homogeneous spaces. J. Math. Soc. Japan 6, 151–159 (1954).

The author proves that the relation

$$\dim G=\dim H+\dim G/H$$

holds for a locally compact group G and any closed subgroup H. The proof relies on the representation of an open subgroup of G as a projective limit of Lie groups. The proof can be shortened using the existence of a local cross-section when G is finite dimensional. [See Mostert, Proc. Amer. Math. Soc. 4, 645–649 (1953); MR 15, 101.]

A. M. Gleason (Cambridge, Mass.).

Goto, Morikuni. Dense imbeddings of locally compact connected groups. Ann. of Math. (2) 61, 154–169 (1955).

A one-one continuous homomorphism φ of a topological group G into a topological group H is called an imbedding of G into H. φ is called a closed or dense imbedding if $\varphi(G)$ is closed or dense in H, respectively. Using the fundamental theorem of Yamabe on the structure of locally compact groups, the author generalizes in the present paper the results of van Est, Malcev, Yamabe and the author himself on the imbedding of locally compact groups. The main theorems are as follows. 1) Let G be a connected locally compact group such that the group of inner automorphisms of G is closed in the group of all automorphisms of G. Then, every imbedding of G into a locally compact group is closed if and only if the center of G is compact. 2) Let φ be a dense imbedding of a connected locally compact group G into a connected locally compact group H satisfying the second countability axiom. Then, there is a one-parameter subgroup X of G such that $H=\overline{\varphi(X)}\varphi(G)$, where $\overline{\varphi(X)}$ denotes the closure of $\varphi(X)$. Several interesting examples of locally compact groups are also given. In particular, it is shown by such an example that the second theorem does not hold if the second countability axiom is not satisfied for H.

K. Iwasawa (Cambridge, Mass.).

Numakura, Katumi. On bicompact semigroups with zero. Bull. Yamagata Univ. (Nat. Sci.) **1951**, no. 4, 405–412 (1951).

Let S denote a topological (Hausdorff) semigroup with zero-element 0 (i.e., $a\cdot 0=0\cdot a=0$, for all $a\in S$). An element $a\in S$ is called nilpotent if $a^n\to 0$. Letting N denote the set of all nilpotent elements of S, a subset of S is called nil if it is contained in N. The radical R of S is the union of all the two-sided nilideals of S, and contains each one-sided nilideal of S. If N is open, then S is called an N-semigroup.

The following structure theorems are given for compact (=bicompact) N-semigroups S. Every non-nil ideal and closed sub-semigroup contains a non-zero idempotent. If S is non-nil, any two-sided non-nil ideal contains a minimal non-nil two-sided ideal K. If R^* denotes the radical of K, then $R^*=R\cap K$, and the difference semigroup $K-R^*$ is completely simple in the sense of D. Rees [Proc. Cambridge Philos. Soc. 36, 387–400 (1940); MR 2, 127]. If e is any non-zero idempotent of K, then eK (Ke) is a minimal right (left) non-nil ideal of S such that the difference semigroup of eK (Ke) modulo its radical contains no proper non-zero right (left) ideal. Moreover, $K=(\bigcup_\lambda G_\lambda)\cup(N\cap K)$, (there is a misprint at this point), where the G_λ are compact, isomorphic and homeomorphic subgroups of S which are disjoint from each other and N. We may write

$$(N\cap K)=(\bigcup_\nu N_\nu)\cup R^*,$$

where the N_ν are nil-semigroups such that $N_\nu^2\subset R^*$. If the idempotents of K commute, then $(N\cap K)=R^*$.

An example is given of a compact semigroup, not an N-semigroup, without minimal non-nil ideals. In another article [Math. J. Okayama Univ. 1, 99–108 (1952); MR 14, 18], the author shows that every compact semigroup T contains a minimal two-sided ideal. The fact that this minimal ideal is (0) in case T has a zero-element motivates in part the present article. This paper contains a large number of misprints, none of which should long delay the reader.

M. Henriksen (Lafayette, Ind.).

Koch, R. J. Remarks on primitive idempotents in compact semigroups with zero. Proc. Amer. Math. Soc. 5, 828–833 (1954).

Throughout this review, X will denote a compact topological (Hausdorff) semigroup with a zero-element (i.e. an element 0 such that $a\cdot 0=0\cdot a=0$ for all a in X). An idempotent e of X is called primitive if e and 0 are the only idempotents in eXe. An element a of X is called nilpotent if 0 is in the closure of the set of positive powers of a. The set of all nilpotent elements of X is denoted by N. Any subset of N is termed nil. A subset A of X is called a bi-ideal if AA and AXA are contained in A.

The main results of this note (which overlap with and improve results of the paper reviewed above) are as follows. Theorem 1: If e is a non-zero idempotent of X, the following are equivalent: (1) e is primitive; (2) $(eXe)\backslash N$ is a group; (3) eXe is a minimal non-nil bi-ideal; (4) Xe is a minimal non-nil left ideal; (5) XeX is a minimal non-nil (two-sided) ideal; (6) every idempotent of XeX is primitive. Theorem 2: If X is connected, then $eXe)\cap N$ is dense in eXe, and $(Xe)\cap N$ is dense in Xe.

M. Henriksen.

NUMBER THEORY

Uhler, H. S. Full values of the first seventeen perfect numbers. Scripta Math. 20 (1954), 240 (1955).

Franqui, Benito, and García, Mariano. 57 new multiply perfect numbers. Scripta Math. 20 (1954), 169–171 (1955).

Gupta, Hansraj. On triangular numbers in arithmetical progression. Math. Student 22, 141–143 (1954).

The general solution of the problem of three triangular numbers, $\frac{1}{2}n(n+1)$, in arithmetic progression is obtained in terms of three arbitrary parameters.

I. Niven.

Leonardi, Raffaele. Some bimagic matrices. Scripta Math. 20 (1954), 165–167 (1955).

***Sierpiński, Wacław. Trójkąty pitagorejskie. [Pythagorean triangles.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. 94 pp. zł. 7.50.

This is an expository account of some dozen problems connected with rational right triangles considered by Diophantus, Fermat, Euler and others. The booklet is intended to stimulate the interest of teachers of elementary mathematics.

D. H. Lehmer (Berkeley, Calif.).

★**Ljunggren, Wilhelm. Ein Satz über die diophantische Gleichung $Ax^2-By^4=C$ ($C=1, 2, 4$).** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 188–194 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Let A, B be positive integers and $C=1$, 2, or 4, such that AB is odd if C is even; A square-free and AB not a perfect square; and let $C=2$ when $A=1$. Further, only such values of A, B, C are considered for which $Ax^2-Bz^2=C$ has a solution, $(x, z)=(a, b)$ being the least positive. If $3+4Bb^2/C$ is not a perfect square, then $Ax^2-By^4=C$ has at most one solution in positive integers x, y. The equation $Ax^2-By^4=4$ has at most one solution in positive relatively prime integers x, y. These are extensions of earlier work by the author [Arch. Math. Naturvid. **41**, no. 10 (1938)]. *I. Niven.*

Val'fiš, A. Z. The additive theory of numbers. XI. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 33–59 (1953). (Russian. Georgian summary)

Suppose $r_k(m, n)$ is the number of solutions in integers $x_1, \cdots, x_k$ of the pair of simultaneous diophantine equations $x_1+\cdots+x_k=m$, $x_1^2+\cdots+x_k^2=n$, where m and n are integers such that $m\equiv n \pmod 2$ and $kn-m^2\geqq 0$. It is known that if $5\leqq k\leqq 8$, then $r_k(m, n)=R_k(m, n)$, where

$$R_k(m, n)=\pi^{\frac{1}{2}k-\frac{1}{2}}k^{1-\frac{1}{2}k}\Gamma(\tfrac{1}{2}k-\tfrac{1}{2})^{-1}(kn-m^2)^{\frac{1}{2}k-\frac{3}{2}}\mathfrak{S}_k(m, n),$$

$\mathfrak{S}_k(m, n)$ being the singular series defined in the paper reviewed below [Kloosterman, Math. Ann. **118**, 319–364 (1942); MR **5**, 33; **9**, 735; Bronkhorst, Dissertation, Univ. of Groningen, 1943; MR **14**, 1063; van der Blij, Nederl. Akad. Wetensch., Proc. **50**, 31–40, 41–48, 166–172, 298–306, 390–396 = Indag. Math. **9**, 16–25, 26–33, 129–135, 188–196, 248–254 (1947); MR **8**, 502]. If $k>8$, the equality $r_k(m, n)=R_k(m, n)$ is no longer generally true, but the present author proves that the weaker result

$$r_k(m, n)=R_k(m, n)+O((n+1)^{\frac{1}{2}k-2}\cdot\log(n+2))$$

still holds. The proof is by induction on k, starting from the case $k=8$. The step from k to $k+1$ is not difficult, and actually most of the paper is devoted to discussing the case $k=8$ in order that the induction can be started. The author seems to feel that Bronkhorst's proof of the equality $r_8(m, n)=R_8(m, n)$ is too complicated and so gives a proof himself on the following lines: (i) He uses an identity of van der Blij in order to express $r_8(m, n)$ in terms of $r_7(\Delta)$, $r_7(\Delta/4)$, and $r_7(\Delta/16)$, where $\Delta=8n-m^2$ and $r_7(u)$ is the number of solutions of $u=y_1^2+\cdots+y_7^2$ in integers $y_1, \cdots, y_7$. (ii) Following van der Blij again, he substitutes the Hardy-Stanley formula [Stanley, J. London Math. Soc. **2**, 91–96 (1927)] for $r_7(u)$ into the results of (i) in order to get a formula for $r_8(m, n)$. (iii) He sums the series $\mathfrak{S}_8(m, n)$ in a more elementary way than Bronkhorst and compares the resulting expression for $R_8(m, n)$ with the result obtained for $r_8(m, n)$ in (ii). Most of the material in this paper is also presented in an expository paper of the author [Uspehi Mat. Nauk (N.S.) **7**, no. 6(52), 97–178 (1952); MR **15**, 936].
P. T. Bateman (Urbana, Ill.).

Lomadze, G. A. On the summation of a singular series. I. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 61–77 (1953). (Russian. Georgian summary)

The singular series occurring in the paper reviewed above is defined as follows

$$\mathfrak{S}_k(m, n)=\sum_{q=1}^{\infty}\sum_{r\mid q}\sum_{a \bmod q}{}'\sum_{b \bmod r}{}'\left(\frac{S(a, q; b, r)}{qr}\right)^k \times\exp 2\pi i\left(-\frac{na}{q}-\frac{mb}{r}\right),$$

where the prime denotes summation over a reduced residue system and

$$S(a, q; b, r)=\sum_{h \bmod qr}\exp 2\pi i\left(\frac{ah^2}{q}+\frac{bh}{r}\right).$$

Kloosterman [Math. Ann. **118**, 319–364 (1942); MR **5**, 33; **9**, 735] has summed this series for all odd squarefree k greater than unity. Using Kloosterman's method, the present author does the same thing for all odd k greater than unity and obtains results similar to those of Kloosterman.
P. T. Bateman (Urbana, Ill.).

Bronštein, B. S. Unboundedness of the sum function of a generalized character. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. 7, 212–220 (1954). (Russian)

The author defines a generalized character as a multiplicative function on the positive integers which takes on only values of absolute value 0 or 1. [This differs somewhat from the definition of Čudakov and Rodosskiĭ, Dokl. Akad. Nauk SSSR (N.S.) **73**, 1137–1139 (1950); MR **12**, 393.] Suppose h is a generalized character which is not a residue-character (in the sense of Dirichlet), but for which there exists a residue-character χ such that $h(p)\neq\chi(p)$ for only finitely many primes p. Then the author shows that $\sum_{n=1}^{x} h(n)$ is unbounded. The proof requires nothing beyond the elementary facts about residue-characters. On the other hand, if h is a generalized character which differs from any residue-character on infinitely many primes, it is possible for $\sum_{n=1}^{x} h(n)$ to be either bounded or unbounded [Čudakov, Uspehi Mat. Nauk (N.S.) **8**, no. 3(55), 149–150 (1953); MR **15**, 289]. *P. T. Bateman* (Urbana, Ill.).

Wright, E. M. A simple proof of a theorem of Landau. Proc. Edinburgh Math. Soc. (2) **9**, 87–90 (1954).

Let $\sigma_k(x)$ be the number of integers $n=p_1\cdots p_k\leqq x$ which are the products of exactly k primes, and let $\pi_k(x)$ be the number of such n for which all the p_i are distinct. The theorem in question asserts that

$$\pi_k(x)\sim\sigma_k(x)\sim\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}.$$

In the present paper this theorem is proved in an elementary way assuming

$$\sum_{p\leqq x}\log p\sim x,\quad \sum_{p<x}\frac{1}{p}\sim\log\log x.$$

L. Carlitz (Durham, N. C.).

Erdös, Paul, and Shapiro, H. N. The existence of a distribution function for an error term related to the Euler function. Canad. J. Math. **7**, 63–75 (1955).

If

$$H(x)=\sum_{n\leqq x}\frac{\phi(n)}{n}-\frac{6}{\pi^2}x,$$

where $\phi(n)$ is the number of positive integers not exceeding n and relatively prime to n, the authors prove that $H(x)$ possesses a continuous distribution function. An essential tool in the proof is the estimate

$$\int_1^x H^2(u)du\sim\frac{x}{2\pi^2}$$

due to the reviewer [Math. Z. **35**, 279–299 (1932)].
S. Chowla (Boulder, Colo.).

Eda, Yoshikazu. On Selberg's function. Proc. Japan Acad. **29**, 418–422 (1953).

The author considers generalizations of Selberg's identity

$$\sum_{\substack{p\leq x\\ p\equiv\lambda(a)}} \log^2 p+\sum_{\substack{pq\leq x\\ pq\equiv\lambda(a)}} \log p\log q=\frac{2}{\varphi(a)}x\log x+O(x),\qquad (\lambda, a)=1,$$

to identities for

$$\sum_{\substack{p_1^{a_1}\cdots p_r^{a_r}\leq x\\ p_1^{a_1}\cdots p_r^{a_r}\equiv\lambda(a)}} \log^{a_1} p_1\cdots\log^{a_r} p_r.$$

Use is made of the umbral calculus of E. T. Bell.

H. N. Shapiro (New York, N. Y.).

Turán, Pal. On the roots of the Riemann zeta function. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 357–368 (1954). (Hungarian)

After a historical survey, without bibliographical details, the author proves the following for $N(\alpha, T)$, the number of roots of $\zeta(\sigma+it)$ with $\sigma\geqq\alpha$, $0<t\leqq T$. Let there be a ϑ with $\frac{1}{2}<\vartheta<1$ such that for $\sigma\geqq\vartheta$, $t\geqq1$ and for some sufficiently small $\eta>0$ there holds

$$|\zeta(\sigma+it)|\leqq c_{10}(\eta)t^{\eta^{1000}}.$$

Then for $\vartheta+4\eta\leqq\sigma_1\leqq1$, $T>c_{11}(\eta)$, there holds

$$N(\sigma_1, T)<c_{12}(\eta)T^{2(1+8\eta)(1-\sigma_1)}\log^5 T.$$

F. V. Atkinson (Ibadan).

Maass, Hans. Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen. Math. Ann. **125** (1952), 235–263 (1953).

The differential operator

$$D=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+(\alpha-\beta)iy\frac{\partial}{\partial x}-(\alpha+\beta)y\frac{\partial}{\partial y}$$

annihilates Eisenstein series of type

$$E(z, w; \alpha, \beta)=\sum_{c, d}\gamma(c, d)(cz+d)^{-\alpha}(cw+d)^{-\beta},$$

where $z=x+iy$, $w=\bar{z}=x-iy$. Let $\{\alpha, \beta\}$ be the linear set of all analytic functions $f(z, w)$, regular in the half-plane $w=\bar{z}$, $\Im(z)>0$, that are annihilated by D. If G is a group of real substitutions $S=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $ad-bc=1$. the author considers automorphic forms, belonging to $\{\alpha, \beta\}$, satisfying

$$f(Sz, Sw)(cz+d)^{-\alpha}(cw+d)^{-\beta}=v(S)f(z, w),\qquad |v(S)|=1,$$

(where v is a set of multipliers for the group G) and which behave in the usual way in the neighborhood of parabolic fix-points of G. These forms are termed to be of type $\{G; \alpha, \beta; v\}$. If $S_0=\begin{pmatrix} 1 & Q\\ 0 & 1\end{pmatrix}\in G$, $v(S_0)=e^{2\pi i\kappa}$ $(0\leqq\kappa<1)$ the functions of type $\{G; \alpha, \beta; v\}$ have a Fourier development

$$f(z, w)=a_0u(y, \alpha+\beta)+b_0+\sum_{n+\kappa\neq0} a_{n+\kappa}W\left(\frac{2\pi|n+\kappa|}{Q}y; \alpha, \beta, \operatorname{sgn}(n+\kappa)\right)e^{2\pi i(n+\kappa)x/Q},$$

where u is an elementary function and W denotes a Whittaker function. If one of the numbers α, β, $\alpha-\beta$ is rational, the Whittaker functions reduce to Bessel functions and exponential functions. The author generalizes the results of Hecke on relations between modular forms and Dirichlet series. If $G=M(Q)$ is the principal congruence subgroup of level Q of the modular group, he also considers the problem of the Euler product development as a generalization of the Hecke-Petersson theory to the theory of the forms $\{M(Q); \alpha, \beta; 1\}$.

H. D. Kloosterman (Leiden).

Maass, Hans. Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen. Math. Ann. **126**, 44–68 (1953).

Let $X=(x_{\mu\nu})$ and $Y=(y_{\mu\nu})$ $(\mu, \nu=1, 2, \cdots, n)$ be $n\times n$ symmetric matrices and let $Z=X+iY=(z_{\mu\nu})$, $W=X-iY=(w_{\mu\nu})$ where the $z_{\mu\nu}$ and $w_{\mu\nu}$ are independent complex variables. The author considers the differential operators defined by

$$D_z=(e_{\mu\nu}\partial/\partial z_{\mu\nu}),\qquad D_w=(e_{\mu\nu}\partial/\partial w_{\mu\nu}),$$
$$K_\alpha=\alpha E+(Z-W)D_z,\quad \Lambda_\beta=-\beta E+(Z-W)D_w$$

and

$$\Omega_{\alpha\beta}=\Lambda_{\beta-\frac{1}{2}(n+1)}K_\alpha+\alpha(\beta-\tfrac{1}{2}n-\tfrac{1}{2})E,$$

where $e_{\mu\nu}=\frac{1}{2}(1+\delta_{\mu\nu})$, $\delta_{\mu\nu}$ is the Kronecker symbol and $E=(\delta_{\mu\nu})$ is the unit matrix. He proves that the Eisenstein series

$$G(Z, W; \alpha, \beta)=\sum_{C, D}\gamma(C, D)|CZ+D|^{-\alpha}|CW+D|^{-\beta}$$

in Siegel's theory of modular forms of order n (C, D run through a complete set of nonassociated coprime pairs of symmetric matrices) satisfy the n^2 differential equations $\Omega_{\alpha\beta}G(Z, W; \alpha, \beta)=0$. The author determines the behavior of the differential operators under symplectic substitutions

$$Z\to(AZ+B)(CZ+D)^{-1},\quad W\to(AW+B)(CW+D)^{-1}.$$

The operator $\Delta=-\operatorname{Trace}(Z-W)((Z-W)D_w)'D_z$ (A' is the transposed matrix of A) is shown to be invariant under symplectic substitutions. This operator is the Laplace-Beltrami operator for the symplectic metric [C. L. Siegel, Amer. J. Math. **65**, 1–86 (1943); MR **4**, 242]. Let $\{\alpha, \beta\}$ be the set of all analytic functions $f(Z, W)$ which are regular in the domain $W=\bar{Z}$ (=complex conjugate of Z), $Y>0$ and which satisfy the differential equations $\Omega_{\alpha\beta}f=0$. If for any symplectic substitution σ the symbol $f(Z, W)|\sigma$ is defined by

$$f(Z, W)|\sigma=|CZ+D|^{-\alpha}|CW+D|^{-\beta}f(\sigma Z, \sigma W),$$

the set $\{\alpha, \beta\}$ is shown to be invariant under symplectic substitutions: $\{\alpha, \beta\}|\sigma=\{\alpha, \beta\}$. The author further constructs certain differential operators M_α and N_β which transform the Eisenstein series $G(Z, W; \alpha, \beta)$ into $\epsilon_n(\alpha)G(Z, W; \alpha+1, \beta-1)$ and $\epsilon_n(\beta)G(Z, W; \alpha-1, \beta+1)$, respectively (the ϵ_n are certain constants). The author surmises that not only the Eisenstein series but even all modular forms f for any group $\mathfrak{G}$ of symplectic substitutions (for which $f(Z, W)\in\{\alpha, \beta\}$ and $f(Z, W)|\sigma=v(\sigma)f(Z, W)$, where $v(\sigma)$ is a certain set of multipliers for $\mathfrak{G}$) show the same behavior under the operators M_α and N_β but he has not been able to prove this conjecture except for $n=1$ and $n=2$ [for $n=1$ cf. the paper reviewed above].

The author finally considers the problem of the Fourier expansion of periodic modular forms of order n. The solution of this problem requires the determination of those functions in $\{\alpha, \beta\}$ which have the form

$$a(Y, T)\exp[i\operatorname{Trace}(TX)],$$

where T is an arbitrary $n\times n$ symmetric real matrix. The linear set $\{\alpha, \beta, T\}$ of functions $a(Y, T)$ is determined by a system of differential equations. This system is completely solved for $n=2$. The maximal number of linearly independent functions in $\{\alpha, \beta, T\}$ is finite.

H. D. Kloosterman (Leiden).

***Wiman, A. Über die Punkte mit ganzzahligen Koordinaten auf gewissen Kurven dritter Ordnung.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 317–323 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The present paper deals with the class of cubic curves

$$(1)\qquad y^2=x^3+px+q \quad (p, q \text{ integers})$$

which are met in triplets of integer points

$$(-(a+b), t), (a, t), (b, t); \quad (-b, s), (-a, s), (a+b, s)$$

by two lines $y=s$, $y=t$, as e.g.:

$$(2)\quad y^2=x^3-7x+10 \qquad (a=1, b=2, s=4, t=2),$$

$$(3)\quad y^2=x^3-172x+820 \qquad (a=2, b=12, s=34, t=22),$$

$$(4)\quad y^2=x^3-172x+505 \qquad (a=2, b=12, s=29, t=13),$$

$$(5)\quad y^2=x^3-112x+2320 \qquad (a=4, b=8, s=52, t=44).$$

For some of them, it may happen that the tangent and chord processes lead to new integer points; for instance, if (x_1, y_1), (x_2, y_2) are any two integer points of the curve (1) (with $x_1 \neq x_2$), their join meets (1) at an integer point if, and only if, x_1-x_2 divides y_1-y_2. It is thus deduced that the curves (3)–(5) contain 24, 60, 58, 70 integer points respectively, in pairs symmetrical with respect to the x-axis. *B. Segre.*

Ayoub, R. G. On Selberg's lemma for algebraic fields. Canad. J. Math. **7**, 138–143 (1955).

A simple proof of Selberg's fundamental lemma in the theory of the distribution of primes has recently been given by Tatuzawa and Iseki [Proc. Japan Acad. **27**, 340–342 (1951); MR **13**, 725] who employed an ingenious application of the Möbius inversion formula. In this paper the author shows that this proof may be extended to yield a corresponding result for algebraic fields. Shapiro [Comm. Pure Appl. Math. **2**, 309–323 (1949); MR **11**, 501] earlier derived the same result using Selberg's methods and deduced as a consequence the prime-ideal theorem.

A. L. Whiteman (Los Angeles, Calif.).

O'Meara, O. T. Characterization of quadratic forms over local fields. Proc. Nat. Acad. Sci. U. S. A. **39**, 969–972 (1953).

Let F be a field complete with respect to a discrete valuation with finite residue class field. Let e be the ordinal of 2. The author gives a complete set of invariants for quadratic forms with integral coefficients in F under integral equivalence, in the cases $e=0$ and $e=1$. These invariants can be expressed in terms of the Hasse symbols and determinants of the principal minors of the matrix of the form. Certain semi-canonical forms, from which the invariants can be read off, are also described. In case $1<e<\infty$, the question of integral equivalence is shown to depend only on the residues of the coefficients mod π^{2e+m+1}, where m is the number of variables. Finally, if $e<\infty$, two forms of unit determinant are integrally equivalent if they are rationally equivalent and represent the same numbers mod π^{2e+1}.

J. Tate (Cambridge, Mass.).

Rédei, Ladislaus. Bedingtes Artinsches Symbol mit Anwendung in der Klassenkörpertheorie. Acta Math. Acad. Sci. Hungar. **4**, 1–29 (1953). (Russian summary)

The author gives an algorithm for constructing a class field of prime-power degree out of cyclic steps of prime degree. This algorithm is based on his Hauptsatz for the symbol appearing in the title. The Hauptsatz, though too complicated to state here, is an elementary theorem about abelian groups of prime-power order (when shorn of its class-field-theoretic garb). *J. Tate.*

Rédei, Ladislaus. Die 2-Ringklassengruppe des quadratischen Zahlkörpers und die Theorie der Pellschen Gleichung. Acta Math. Acad. Sci. Hungar. **4**, 31–87 (1953). (Russian summary)

Let $\Omega=Q(\sqrt{d})$, d square-free, be a quadratic field. Let m be an odd square-free natural number prime to d. Let H be the group of ideals prime to m, some power of which is principal with a generator which is congruent to a rational number mod mp_∞. Using the method of the paper reviewed above, the author discusses in great detail the explicit determination of the corresponding class group and construction of the corresponding class field. He also treats thoroughly the theory of the negative Pell equation $x^2-dm^2y^2=-1$ and of certain diophantine equations of the form

$$d_1m_1^2x_1^2-d_2m_2^2x_2^2=z^{2^n} \quad (\text{resp., } =1),$$

which is intimately related to the class-field problem. Some special cases, and some numerical examples are given as illustrations of the theory. *J. Tate.*

Fogel' [Fogelis], È. K. A finite proof of the Gauss-Dirichlet formula. Latvijas PSR Zinātņu Akad. Vēstis **1950**, no. 9 (38), 117–125 (1950). (Russian. Latvian summary)

This is an alternative version of the author's "finite" proof of the well known (finite) formulae for the class number $h(d)$ of binary quadratic forms of negative discriminant d [cf. Latvijas PSR Zinātņu Akad. Fiz. Inst. Raksti **3**, 49–63 (1952); MR **16**, 222]. The logical status of this proof is easier to understand, since it involves none of the "semi-finite" analysis of the other version. There is no differentiation or integration; $\cos x$, $\sin x$, π are replaced by rational approximations based on the power series for cos and sin; and classical Fourier analysis is approached by way of finite Fourier analysis of periodic functions of an integral variable.

A. E. Ingham (Cambridge, England).

Linnik, Yu. V., and Malyšev, A. V. Applications of the arithmetic of quaternions to the theory of ternary quadratic forms and to the decomposition of numbers into cubes. Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 3–71 (1953); corrections, **10**, no. 1 (63), 243–244 (1955). (Russian)

This is a semi-expository paper in which the authors give a connected presentation, with full details of proof, of results on positive ternary quadratic forms which they have published (mostly without complete proofs) during the last fifteen years [Izv. Akad. Nauk SSSR. Ser. Mat. **4**, 363–402 (1940); Dokl. Akad. Nauk SSSR (N.S.) **87**, 175–178 (1952); **89**, 209–211, 405–406 (1953); MR **2**, 348; **15**, 406]. The positive ternary quadratic forms studied are those properly primitive forms f with integral coefficients and invariants $[\Omega, \Delta]$ such that Ω is odd, $\Delta=1$, and $(f|p)=(-1)^{(p-1)/2}$ for each prime p dividing Ω. The reciprocal forms to these are also considered. Such forms are singled out because the representations of large positive integers by them can be studied by means of the arithmetic of quaternions, and also because they include important special cases, such as those used by Linnik in proving that every sufficiently large positive integer can be expressed as a sum of seven non-negative integral cubes [Mat. Sb. N.S. **12**(**54**), 218–224 (1943); MR **5**, 142]. The proof of this last fact is also given in full in the present paper, and apparently one of the purposes of the paper is to put this proof on a sound footing. However,

Unfortunately, an error in Linnik's work which was pointed out by G. Pall [MR 2, 348, Amer. J. of Math. 64, 503-513 (1942); MR 4, 34] is perpetuated in the prev. paper, the correction straightens out the difficulty.

G. L. Watson's proof of the same result [J. London Math. Soc. 26, 153–156 (1951); MR 13, 915] is much shorter and requires less preparation. *P. T. Bateman.*

Swinnerton-Dyer, H. P. F. The inhomogeneous minima of complex cubic norm forms. Proc. Cambridge Philos. Soc. 50, 209–219 (1954).

Let K_1, K_2, K_3 be conjugate cubic algebraic number fields (K_1 real, K_2, K_3 complex conjugate). Let $\omega_{11}, \omega_{12}, \omega_{13}$ be a basis for the integers of K_1, $\omega_{i1}, \omega_{i2}, \omega_{i3}$ the conjugate basis for the integers of K_i and $\xi_i=\omega_{i1}x_1+\omega_{i2}x_2+\omega_{i3}x_3$. Let $M(K; x_1', x_2', x_3')=\min|\xi_1\xi_2\xi_3|$ taken over all points

$$(x_1, x_2, x_3)\equiv(x_1', x_2', x_3') \pmod 1$$

and $M(K)=\max M(K; x_1', x_2', x_3')$ taken over all points (x_1', x_2', x_3') in the unit cube $0\leqq x_j'<1$. Let d be the discriminant of K_1, K_2 and K_3. The author shows that if $|d|<1237$ then $M(K)<|d|^{2/3}/16\sqrt[3]{2}$, and that this result is best possible, in the sense that neither the exponent $\frac{2}{3}$ nor the numerical constant can be improved.

H. S. A. Potter (Aberdeen).

Lursmanašvili, A. P. On the number of lattice points in multidimensional spheres. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 79–120 (1953). (Russian. Georgian summary)

Let $A_k(x)$ be the number of lattice points in the k-dimensional sphere $z_1^2+z_2^2+\cdots+z_k^2\leqq x$. Write

$$A_k(x)=V_k(z)+\mathrm{P}_k(x),$$

where $V_k(x)$ is the content of the sphere. The author obtains the results

$$\mathrm{P}_{2k}(x)=\pi^k\{(1-2^{-k})\zeta(k)\Gamma(k)\}^{-1}\sum_{s=1}^{m}\Psi_{k,s}(x)x^{k-s}+O(x^{k-m-\frac{1}{2}}),$$

when k is even, and

$$\mathrm{P}_{2k}(x)=\pi^k\{L(k)\Gamma(k)\}^{-1}\sum_{s=1}^{m}\Phi_{k,s}(x)x^{k-s}+O(x^{k-m-\frac{1}{2}}),$$

when k is odd. Here $k\geqq4$ and $\Psi_{k,s}(x)$ and $\Phi_{k,s}(x)$ are certain complicated combinations of functions such as the Bernoullian polynomials. The proofs, which are too complicated to describe here, follow the methods developed by A. Walfisz. [See also MR 15, 941 where this paper is referred to.]

R. A. Rankin (Glasgow).

Linnik, Yu. V. The asymptotic distribution of lattice points on a sphere. Dokl. Akad. Nauk SSSR (N.S.) 96, 909–912 (1954). (Russian)

Let Γ be a convex cone with center at the origin in three-dimensional Euclidean space and suppose that the part of the unit-sphere which is contained in Γ has area $4\pi\lambda$. If m is a positive integer greater than unity, let $R(m)$ denote the number of primitive lattice points (x, y, z) on the sphere $x^2+y^2+z^2=m$ and let $R_\Gamma(m)$ denote the number of these lattice points which also lie in Γ. It is known that $R(m)=0$ if $m\equiv0$, 4, 7 (mod 8), $R(m)=8h(-m)$ if $m\equiv3$ (mod 8), and $R(m)=12h(-m)$ if $m\equiv1$, 2, 5, 6 (mod 8), where $h(-m)$ is the number of primitive classes of binary quadratic forms $au^2+buv+cv^2$ (with integral coefficients) such that $b^2-4ac=-4m$ [see, e.g., p. 99 of a paper of the reviewer, Trans. Amer. Math. Soc. 71, 70–101 (1951) [MR 13, 111] in connection with pp. 127 and 152 of Landau's Vorlesungen über Zahlentheorie, vol. 1, Hirzel, Leipzig, 1927]. Thus by the Heilbronn-Siegel theorem $R(m)$ tends to infinity if m goes to infinity through the integers congruent to 1, 2, 3, 5, or 6 modulo 8. The author sketches a proof that if, in addition to satisfying the restriction $m\equiv1$, 2, 3, 5, 6 (mod 8), m is required to be a quadratic residue of some fixed odd prime number q, then $\lim_{m\to\infty}R_\Gamma(m)/R(m)=\lambda$. This is an improvement of an earlier result of Linnik and Malyšev [Dokl. Akad. Nauk SSSR (N.S.) 89, 209–211 (1953); MR 15, 406] to the effect that $\lim\inf_{m\to\infty}R_\Gamma(m)/R(m)>0$ under the same restrictions on m. The proof uses methods from a paper of Malyšev [ibid. 93, 771–774 (1953); 95, 700 (1954); MR 15, 936], elementary considerations from probability theory, and a geometrical lemma ascribed to V. A. Zalgaller.

P. T. Bateman (Urbana, Ill.).

HAVE

***Keller, Ott-Heinrich. Geometrie der Zahlen.** Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. I 2, 27. Band I. Algebra und Zahlentheorie. 2. Teil. C. Reine Zahlentheorie. Heft 11, Teil III. B. G. Teubner Verlagsgesellschaft, Leipzig, 1954. 84 pp. DM 8.80.

This is an account of results in the theory of numbers obtained by geometric methods up to 1951. Among the chapter headings are convex bodies in lattices, star bodies, linear forms, minima of homogeneous forms, inhomogeneous forms, definite quadratic forms, continued fractions and algebraic numbers. Some proofs of fundamental results are sketched. The results on star bodies which have been obtained in the last two decades are listed with considerable completeness. Included among these is a paragraph of conjectures made by Mahler. The chapter on definite quadratic forms contains a clear account of the work done on the reduction problem. *D. Derry.*

Kogoniya, P. G. On the structure of the set of Markov numbers. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 121–133 (1953). (Russian. Georgian summary)

Let α be any irrational number satisfying $0<\alpha<1$ and with partial quotients $a_1, a_2, \cdots$, so that α has the continued fraction representation $[0, a_1, a_2, \cdots]$. Let $L(\alpha)$ be inf c subject to the condition that $|\alpha-p/q|<c/q^2$ has infinitely many solutions for integers p, q with $q>0$. The set of all positive $L(\alpha)$ is denoted by M_L. In the range $L(\alpha)>\frac{1}{3}$, M_L contains only a countable infinity of values which form a decreasing sequence, decreasing from $1/\sqrt{5}$ to the limit $\frac{1}{3}$. The author is chiefly concerned with that part of M_L in the range $0<L(\alpha)<\frac{1}{3}$. If $M(N)$ denotes the set of all α for which $\lim\sup a_k=N$ and $M_L(N)$ is the set of all $L(\alpha)$ corresponding to α in $M(N)$, then M_L is the union of all sets $M_L(N)$ for $N=1, 2, 3, \cdots$. It is proved that, for $N\geqq2$, $M_L(N)$, and consequently M_L, has the power of the continuum. This is done by constructing a set of numbers $L(\alpha)$ of this power belonging to $M_L(N)$, the construction being easier for $N\geqq4$ than for $N=2$ and 3.

It is also shown that $\frac{1}{3}$ is the maximum point of condensation of the set M_L; in fact, the set of all points of $M_L(2)$ lying in the interval $1/(3+\epsilon)<L(\alpha)<1/3$ has the power of the continuum for every $\epsilon>0$. Finally it is shown that the set M_L has Hausdorff dimension 1. The proof uses Jarnik's inequalities for the Hausdorff dimension of the set of α for which $a_k\leqq N$ for all k. *R. A. Rankin* (Glasgow).

Günther, Alfred. Über transzendente 𝔭-adische Zahlen. II. Zur approximation transzendenter 𝔭-adischer Zahlen durch rationale. J. Reine Angew. Math. 193, 1–10 (1954).

This is a continuation of an earlier paper [same J. 192, 155–156 (1953); MR 15, 604]. The author establishes two

theorems related to a theorem of A. Gelfond [Mat. Sb. N.S. 7(49), 7–25 (1940); MR 1, 292]. (1) Let α be an algebraic $\mathfrak{p}$-adic number with $0<|\alpha-1|_\mathfrak{p}<p^{-1/(p-1)}$, β a $\mathfrak{p}$-adic number with $0<|\beta-1|_\mathfrak{p}<p^{-1/(p-1)}$, and $\eta=\log\alpha/\log\beta$ an irrational algebraic $\mathfrak{p}$-adic integer. Then for every $\epsilon>0$ the inequality

$$(*)\qquad \left|\beta-\frac{q_1}{q_2}\right|_\mathfrak{p}<p^{-(\log q)^{2+\epsilon}/\epsilon}$$

has at most a finite number of solutions in relatively prime rational integers q_1, q_2 with $q=\max\{|q_1|, |q_2|\}$. (2) Let α be a $\mathfrak{p}$-adic number with $0<|\alpha-1|_\mathfrak{p}<p^{-1/(p-1)}$, β an algebraic $\mathfrak{p}$-adic number with $0<|\beta-1|_\mathfrak{p}<p^{-1/(p-1)}$, and $\eta=\log\alpha/\log\beta$ an irrational algebraic $\mathfrak{p}$-adic integer. Then for every $\epsilon>0$ the inequality

$$\left|\alpha-\frac{q_1}{q_2}\right|_\mathfrak{p}p^{-(\log q)^{2+\epsilon}/\epsilon}$$

has at most a finite number of solutions in relatively prime rational integers q_1, q_2 with $q=\max\{|q_1|, |q_2|\}$. From these theorems, the following corollaries are deduced. (1.1) If η is an irrational algebraic $\mathfrak{p}$-adic integer and β a $\mathfrak{p}$-adic number with $0<|\beta-1|_\mathfrak{p}<p^{-1/(p-1)}$, and if for every $\epsilon>0$ the inequality (*) has an infinite number of solutions in relatively prime rational integers q_1, q_2, with $q=\max\{|q_1|, |q_2|\}$, then β^η is transcendental. (2.1) If β is an algebraic $\mathfrak{p}$-adic number with $0<|\beta-1|_\mathfrak{p}<p^{-1/(p-1)}$ and if η is an irrational algebraic $\mathfrak{p}$-adic integer, then for every $\epsilon>0$ the inequality

$$\left|\beta^\eta-\frac{q_1}{q_2}\right|_\mathfrak{p}<p^{-(\log q)^{2+\epsilon}/\epsilon}$$

has at most a finite number of solutions in relatively prime rational integers q_1, q_2 with $q=\max\{|q_1|, |q_2|\}$.

M. Newman (Washington, D. C.).

★**Jarník, Vojtěch. Über lineare diophantische Approximationen.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 189–192. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

Summary of the following two papers: Apfelbeck, Czechoslovak Math. J. 1(76), 119–148 (1952) [MR 14, 359]; Kurzweil, ibid. 1(76), 149–178 (1952) [MR 14, 454].

Beatty, S. Elementary proof that e is not quadratically algebraic. Amer. Math. Monthly 62, 32–33 (1955).

ANALYSIS

Popoviciu, Tiberiu. On the mean-value theorem for continuous functions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 353–356 (1954). (Hungarian)

Let real numbers $x_1<x_2<\cdots$ be assigned. If $f(x)$ is a real-valued function defined for the arguments to be used, then one defines the difference quotient $[x_1, \cdots, x_{n+1}; f]$ of order n recursively by the formula

$$[x_1, \cdots, x_{n+1}; f]=\frac{[x_2, \cdots, x_{n+1}; f]-[x_1, \cdots, x_n; f]}{x_{n+1}-x_1},$$

starting with $[x_1; f]=f(x_1)$. The purpose of the note is to state, without proof, a series of elementary theorems and identities involving such difference quotients. In the way of motivation, the author points out that the remainders appearing in many formulas relating to interpolation and numerical evaluation are sums of difference quotients of the type described above. *T. Radó* (Columbus, Ohio).

Carleson, Lennart. A proof of an inequality of Carleman. Proc. Amer. Math. Soc. 5, 932–933 (1954).

Let $m(x)$ be a continuous convex function for $x\geqq 0$, with $m(0)=0$, and let $-1<p<\infty$. It is proved that

$$\int_0^\infty x^p e^{-m(x)/x}dx\leqq e^{p+1}\int_0^\infty x^p e^{-m'(x)}dx,$$

and that the constant e^{p+1} cannot be improved. The case $p=0$ is used to give a proof of Carleman's inequality; i.e., if $\sum_{n=1}^\infty a_n$ is a convergent series with positive terms, then $\sum_{n=1}^\infty (a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^\infty a_n$. *F. F. Bonsall.*

Reid, William T. A note on the Hamburger and Stieltjes moment problems. Proc. Amer. Math. Soc. 5, 521–525 (1954).

The following result is proved: The Hamburger [the Stieltjes] moment problem $\mu_n=\int_{-\infty}^\infty t^n d\alpha(t)$ $[\mu_n=\int_0^\infty t^n d\alpha(t)]$ has a nondecreasing solution $\alpha(t)$ with infinitely many points of increase, provided the following condition (*) [the following conditions (*) and (**)] is [are] satisfied:

$$(*)\qquad \mu_0\geqq a_0,\quad \mu_{2n}\geqq a_n+(a_{n-1}-a_n)^{-1}\sum_{j=0}^{n-1}\mu^2_{n+j}\quad (n=1, 2, \cdots),$$

$$(**)\qquad \mu_1\geqq b_0,\quad \mu_{2n+1}\geqq b_n+(b_{n-1}-b_n)^{-1}\sum_{j=0}^{n-1}\mu^2_{1+n+j}\quad (n=1, 2, \cdots),$$

where $\{a_n\}$ $[\{b_n\}]$ is a monotone decreasing sequence of positive numbers. Moreover, if $\lim a_n>0$ [if $\lim a_n>0$ and $\lim b_n>0$], then the moment problem is indeterminate. This theorem contains as a special case a previous result of Boas [D. V. Widder, The Laplace transform, Princeton, 1941, pp. 140–142; MR 3, 232]. *M. Cotlar* (Mendoza).

Popoviciu, Tiberiu. Sur le reste dans quelques formules de dérivation numérique. I. Quelques propriétés des formules de dérivation numérique d'exactité maximum. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 53–122 (1952). (Romanian. Russian and French summaries)

The formulas which the author considers are of the form

$$(1)\qquad f^{(m+r)}(x_0)=\sum_{j=0}^{r-1}a_jf^{(j)}(x_0)+\sum_{i=1}^{s}\sum_{j=0}^{r_i-1}a_{i,j}f^{(j)}(x_i),$$

with a remainder R. The order is $r-1+\sum(r_i-1)+s$ $(+1$ if $r>0)$. The degree of exactitude is the smallest integer n such that $R=0$ for all polynomials of degree n but $R\neq 0$ for some polynomial of degree $n+1$. The author is chiefly concerned with the formula (E) obtained by taking the right-hand side of (1) to be the Lagrange-Hermite interpolating polynomial of $f(x)$ with nodes x_0 and x_i, each repeated according to its multiplicity. He shows that (E) has the largest degree of exactitude of all formulas (1) with the same r and r_i. Generally speaking, the order exceeds the degree of exactitude by 1; otherwise these quantities are equal and the formula is called exceptional. The author gives a number of theorems on this situation. Formula (E) is called reducible if, when written in the form (1), it does not involve any values of $f(x)$ (but only of its

derivatives). The remainder R is said to be of simple form if it can be put in the form of a constant M times an $(n+2)$-point divided difference of $f(x)$. This is shown to happen if and only if $R\neq 0$ for all $f(x)$ which are convex of order n. Necessary and sufficient conditions are given for the remainder to be of simple form when the degree of exactitude is -1, 0 or 1. A number of sufficient conditions are given in higher cases. The author lays particular stress on the result that (E) has a remainder of simple form if the nodes are symmetric with respect to x_0. There is detailed discussion of the case $m=1$. Finally the author shows how to write out a formula (E) explicitly, and writes out explicitly the 76 reducible, exceptional and symmetric formulas whose degrees of exactitude are at most 5. *R. P. Boas, Jr.*

Malliavin, Paul. La quasi-analyticité généralisée sur un intervalle borné. C. R. Acad. Sci. Paris **240**, 41–42 (1955).

The author announces some results on the problem of when $|f^{(n)}(x)|\leqq M_n$ $(0\leqq x\leqq R)$ and $f^{(n)}(0)=0$ for a sequence of n's imply that $f(x)$ is analytic in a circle around 0.
R. P. Boas, Jr. (Evanston, Ill.).

Nikol'skiĭ, S. M. Properties of certain classes of functions of several variables on differentiable manifolds. Mat. Sb. N.S. **33**(75), 261–326 (1953). (Russian)

In this paper the author further develops the theory of functions in his classes $H_p^{(r_1,\cdots,r_n)}(M_1,\cdots,M_n)$ [Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); MR **14**, 32]. These functions are defined throughout Euclidean n-space R^n, but by means of inequalities analogous to the ones used in defining $H_p^{(r_1,\cdots,r_n)}(M_1,\cdots,M_n)$ the author now introduces corresponding classes $H_p^{(r_1,\cdots,r_n)}(G;M_1,\cdots,M_n)$ of functions on a region $G\subset R^n$. A rather extensive study is made of the behavior of such functions on differentiable manifolds of the form

$$x_{m+1}=\varphi_{m+1}(x_1,\cdots,x_m)$$
$$\cdots$$
$$x_n=\varphi_n(x_1,\cdots,x_m).$$

The results are too detailed to be incorporated here, but the trend of the paper is indicated by the following theorem, the converse of which corresponds to the case of $p'=p$ in the author's embedding theorem (Theorem 12 of the above-cited work). Theorem: let $\lambda=(\lambda_{m+1},\cdots,\lambda_n)$ be a vector of positive integral components, let $\rho_1^{(\lambda)},\cdots,\rho_m^{(\lambda)}$ satisfy

$$\rho_i^{(\lambda)}=r_i\left(1-\sum_{j=m+1}^{n}\frac{\lambda_j}{r_j}-\frac{1}{p}\sum_{j=m+1}^{n}\frac{1}{r_j}\right)>0,$$

and let $\varphi_{(\lambda)}(x_1,\cdots,x_m)$ be a function in $H_p^{(\rho)}$, where $(\rho)=(\rho_1^{(\lambda)},\cdots,\rho_m^{(\lambda)})$; then there exists a function $f(x_1,\cdots,x_n)$ in $H_p^{(r_1,\cdots,r_n)}$ such that

$$\frac{\partial^{\lambda_{m+1}+\cdots+\lambda_n}f}{\partial x_{m+1}^{\lambda_{m+1}}\cdots\partial x_n^{\lambda_n}}=\varphi_{(\lambda)}(x_1,\cdots,x_m)$$

holds for $x_{m+1}=\cdots=x_n=0$. Similar theorems are given for functions on differentiable manifolds, the role of the partial derivatives being taken over by derivatives in $n-m$ mutually orthogonal directions normal to the manifold. Inequalities connecting the norm of the derived function with that of the given function are obtained, and reference is made to other papers in which these inequalities are applied to boundary-value problems [Nikol'skiĭ, Dokl. Akad. Nauk SSSR (N.S.) **83**, 23–25; **84**, 652 (1952); **88**, 409–411 (1953); MR **13**, 943; **15**, 425; Amanov, ibid. **88**, 389–392 (1953); MR **15**, 124]. *M. G. Arsove* (Seattle, Wash.).

Nyikoljszkij, Sz. M. Properties of certain classes of functions of several variables defined on differentiable manifolds and their application to variational problems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 243–252 (1953). (Hungarian)

An introductory lecture outlining results of which the author has since published a detailed account [see the preceding review]; an application to boundary problems of Dirichlet type is explained. *F. V. Atkinson.*

Kudryavcev, L. D. On a generalization of S. M. Nikol'skiĭ's theorem on the compactness of classes of differentiable functions. Uspehi Mat. Nauk (N.S.) **9**, no. 1(59), 111–120 (1954). (Russian)

Typical of the generalized theorems (of which there are three) is the following. Let E^n be Euclidean n-space, $L_p^{(n)}$ the space of real-valued functions of summable pth power on E^n $(0<p\leqq\infty)$, and $H_p^{(r_1,\cdots,r_n)}(M_1,\cdots,M_n)$ the Nikol'skiĭ class [all functions $f\in L_p^{(n)}$ having generalized partial derivatives $f_{x_i}^{(\bar r_i)}\in L_p^{(n)}$ satisfying $\|\Delta_{x_i}^{(2)}(f_{x_i}^{(\bar r_i)},h)\|_p^{(n)}\leqq M_i|h|^{\alpha_i}$, where $M_i\geqq 0$, $r_i>0$, and $r_i=\bar r_i+\alpha_i$ for $\bar r_i$ an integer and $0<\alpha_i\leqq 1$ $(i=1,2,\cdots,n)$]. Let $1\leqq m\leqq n$, and for $R>0$ let

$$Q_R^{(m)}(x_{m+1}^{(0)},\cdots,x_n^{(0)})=\{(x_1,\cdots,x_n):|x_i|\leqq R$$
$$(i=1,2,\cdots,n);\quad x_{m+1}=x_{m+1}^{(0)},\cdots,x_n=x_n^{(0)}\}.$$

If $1-p^{-1}\sum_{m+1}^n r_i^{-1}>0$ and $\{f_k\}$ is a sequence of functions in $H_p^{(r_1,\cdots,r_n)}(M_1,\cdots,M_n)$ bounded in $L_p^{(n)}$, then there is a subsequence $\{f_{k_s}\}$ such that for every choice of $x_{m+1}^{(0)},\cdots,x_n^{(0)}$ and R the sequence

$$\{f_{k_s}(x_1,\cdots,x_m,x_{m+1}^{(0)},\cdots,x_n^{(0)})\}$$

converges in the $L_p^{(m)}$ sense on $Q_R^{(m)}(x_{m+1}^{(0)},\cdots,x_n^{(0)})$, the convergence being uniform with respect to $x_{m+1}^{(0)},\cdots,x_n^{(0)}$. Without the uniformity conclusion, this result is essentially Theorem 14 of Nikol'skiĭ [Trudy Mat. Inst. Steklov. **38**, 244–278 (1951); MR **14**, 32]. *M. G. Arsove.*

Calculus

Gonçalves, J. Vicente. Sur l'égalité des dérivées partielles similaires. Univ. Lisboa. Revista Fac. Ci. A. (2) **3**, 161–170 (1954).

The author uses the following lemma in order to discuss the values of the coefficients in the finite Taylor series of a function $y=f(x)$. If $a_0,\cdots,a_{n-1},b_0,\cdots,b_{n-1}$ are constants, and, in the interior of $(0,\epsilon)$, we have

$$a_0+\cdots+a_{n-1}x^{n-1}+A_nx^n=b_0+\cdots+b_{n-1}x^{n-1}+B_nx^n$$

with $\lim_{x\to 0}xA_n=\lim_{x\to 0}xB_n=0$, then $a_0=b_0,\cdots,a_{n-1}=b_{n-1}$, $A_n(x)=B_n(x)$. *E. F. Beckenbach* (Los Angeles, Calif.).

Gonçalves, J. Vicente. Sur la $n^{\text{ième}}$ formule de Taylor. Univ. Lisboa. Revista Fac. Ci. A. (2) **3**, 187–190 (1954).

The author shows that if $f_y(x,0)$ and $f_{xy}(0,0)$ exist, and $f_{x^2y}(x,y)$ remains bounded in the neighborhood of the origin, then $f_{yx}(0,0)$ also exists and satisfies $f_{yx}(0,0)=f_{xy}(0,0)$. A similar result is given for higher-order partial derivatives.
E. F. Beckenbach (Los Angeles, Calif.).

Cerrillo, Manuel V. On the evaluation of integrals of the type

$$f(\tau_1, \tau_2, \cdots, \tau_n) = \frac{1}{2\pi i}\int F(s)e^{W(s,\tau_1,\tau_2,\cdots,\tau_n)}ds$$

and the mechanism of formation of transient phenomena. 2a. Elementary introduction to the theory of the saddle-point method of integration. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. No. **55 : 2a** (1950), iv+74 pp. (1954).

This report contains a detailed and elementary exposition of the application of the method of steepest descents and similar methods to the approximate evaluation of integrals of the form

$$f(t) = \frac{1}{2\pi i}\int_{\gamma_0} F(s)e^{W(s,t)}ds,$$

where $W(s, t) = st - \phi(s)$ and $F(s)$ "is supposed to be free from terms of exponential behavior". Throughout the report, emphasis is placed on the technique of obtaining the approximations in a form useful in circuit theory. All proofs and analyses of error are postponed, and will be contained in future reports of this series. *A. Erdélyi.*

Theory of Sets, Theory of Functions of Real Variables

Have

★Bourbaki, N. Eléments de mathématique. XVII. Première partie : Les structures fondamentales de l'analyse. Livre I : Théorie des ensembles. Chapitre I : Description de la mathématique formelle. Chapitre II : Théorie des ensembles. Actualités Sci. Ind., no. 1212. Hermann & Cie, Paris, 1954. 136 pp. 1500 francs.

The volume under review is the most recently published part of the comprehensive treatise that the Bourbaki group has been writing for the last sixteen years; it consists of the first two chapters of the first book of the treatise.

An important virtue of the work is that it exists. It is the first book, in any language, whose aim is to give a systematic development of axiomatic set theory, and, as such, it takes an appreciable step toward filling a deplorable gap in the literature. The expert, to be sure, will not find any startlingly new theorems or techniques. The student, however, especially if he already has a good grasp of the so-called naive approach to set theory, will find the book a big help in putting his mathematical knowledge on a foundation much more likely to be secure than any that unbridled intuition can supply.

In several matters of notation, terminology, emphasis, and organization the authors have made some arbitrary choices different from those of any other worker in the field, and the treatment exhibits an unusual compromise between frugality and extravagance with respect to symbols and axioms. For these reasons, the reviewer's (admittedly subjective) impression is that the work has a somewhat "ad hoc" character.

Chapter I describes the first-order functional calculus with equality, and Chapter II presents the so-called Zermelo-Fraenkel axioms for set theory. This summary of the contents is accurate in broad outline but not in detail. The remainder of this review is devoted to a more accurate description of the contents and, along the way, of some of those technical aspects of the approach that have not been universally adopted so far.

Chapter I uses four basic logical symbols (namely, $\vee$, $\neg$, τ, and $\square$) together with the connecting lines whose use is exemplified below. The first two of these symbols denote disjunction and negation respectively, and the third is the Hilbert ε symbol. The effect of $\square$ is to eliminate all variables (free or bound) from the unabbreviated versions of all theorems. The following example (which makes use of the sign of equality, introduced slightly later) illustrates one step in such an elimination: the term $\tau_x(=xy)$, which Hilbert would have written as $\varepsilon_x(x=y)$, has for its completely formal version $\overline{\tau = \square} y$.

A well-formed formula of the theory is called a "relation". After defining "theorem" as a provable relation, in essentially the usual way, the authors make an unusual comment concerning this concept. The comment, in full, reads as follows: "Cette notion est donc essentiellement relative à l'état de la théorie considérée, au moment où on la décrit: une relation d'une théorie $\mathcal{T}$ *devient* un théorème de $\mathcal{T}$ lorsqu'on a réussi à l'insérer dans une démonstration de $\mathcal{T}$. Dire qu'une relation de $\mathcal{T}$ 'n'est pas un théorème de $\mathcal{T}$' ne peut avoir de sens si on ne précise pas le stage du développement de $\mathcal{T}$ auquel se réfère."

The first-order functional calculus (with equality) is described by seven axiom schemata. The first four of these are the familiar axioms for the propositional calculus. The fifth is the equally familiar $p(a) \to (\exists x)p(x)$, where $(\exists x)p(x)$ denotes $p(\tau_x(p(x)))$. The sixth and the seventh introduce the equal sign and establish its connection with the basic logical symbols; they are

$$(a=b) \to (p(a) \leftrightarrow p(b))$$

and

$$(\forall x)(p(x) \leftrightarrow q(x)) \to (\tau_x(p(x)) = \tau_x(q(x)))$$

respectively. (These three schemata are written here in conventional notation, except for the use of τ in place of ε.)

Between the two chapters of the volume there is a brief digression (in the form of an appendix) whose purpose is to characterize the terms and relations of a mathematical theory by means of certain concepts pertinent to the theory of free semigroups.

Chapter II introduces two specifically set-theoretic symbols, namely ϵ (for belonging) and $\supset$ (for ordered pairs). The symbol $\supset$ is to be used in formal contexts such as $\supset ab$, where a and b are terms. (The informal symbol for the resulting term is (a, b), the ordered pair of a and b. The Kuratowski definition, $(a, b) = \{\{a\}, \{a, b\}\}$, is given in an exercise.) Since the formal symbol occurs in the volume exactly four times, since, presumably, it will never be used again, and since omitting it and the axiom concerning it does not alter the power of the theory at all, the use of $\supset$ is the most outstanding example of what was called symbolic extravagance above.

Set theory is described by one new axiom schema and five particular axioms. Most of these are stated in terms of the abbreviation $\mathrm{Coll}_x\, p(x)$ for the relation

$$(\exists y)(\forall x)((x \in y) \leftrightarrow p(x)),$$

where $p(x)$ is a relation with no free occurrences of y. The schema is a slight generalization of a combination "Aussonderung"-union-substitution axiom: it takes the form

$$(\forall y)(\exists X)(\forall x)(p(x, y) \to (x \in X))$$
$$\to (\forall Y)\, \mathrm{Coll}_x\, ((\exists y)((y \in Y) \,\&\, p(x, y))),$$

with appropriate stipulations designed to avoid unintended alphabetic collisions. The first four axioms are the axiom of extension, the axiom of unordered pairs, the axiom of

ordered pairs (if $(x, y) = (x', y')$, then $x = x'$ and $y = y'$), and the axiom of the power set. The fifth axiom is not discussed in this volume; according to a summary of the axioms at the end of the volume, it states the existence of an infinite set.

Many of the elementary concepts of set theory are treated in exhaustive detail. Among these concepts are: sets of ordered pairs (often called "relations" in English, here they are called "graphs"), functions (a one-to-one correspondence from one set onto another is called a "bijection"), unions, intersections, complements, Cartesian products, and equivalence relations.

There are several interesting and amusing exercises. One of the most striking ones challenges the reader to detect the error (based on a popular set-theoretic solecism) in a "proof" that Fermat's last theorem is false. *P. R. Halmos.*

Fodor, G. On a problem in set theory. Acta Sci. Math. Szeged **15**, 240–242 (1954).

Suppose that M is an infinite set of power $\mathfrak{m}$, $\mathfrak{n} < \mathfrak{m}$, and with every $m \in M$ there is associated a subset M_m of M such that $|M_m| < \mathfrak{n}$. Then there exists a subset M' of M, with $|M'| = \mathfrak{m}$, such that $|M - \bigcup_{m \in M'} M_m| = \mathfrak{m}$. This was proved by Erdös [Michigan Math. J. **2**, 51–57 (1954); MR **16**, 20] using the generalized continuum hypothesis, and is proved without the use of this hypothesis in the present paper. *F. Bagemihl* (Princeton, N. J.).

Sudan, Gabriel. Les discontinus dyadiques et la puissance du continu. Acad. Repub. Pop. Romåne. Stud. Cerc. Mat. **2**, 364–386 (1951). (Romanian. Russian and French summaries)

Some familiar sets having the power of the continuum are shown to have this power by demonstrating that they possess subsets that are dyadic discontinua.

F. Bagemihl (Princeton, N. J.).

Bagemihl, F. A note on Scheeffer's theorem. Michigan Math. J. **2** (1953–54), 149–150 (1955).

A theorem of L. Scheeffer [Acta Math. **5**, 279–296 (1884)] concerning subsets of the real line asserts that if the set E is countable and the set N is nowhere dense, then there exists an everywhere dense set D such that for every $d \in D$ the intersection with N of the d-translate $\{e+d : e \in E\}$ of E is empty. The author notes that in this result one may replace "nowhere dense" by "of first category" and "everywhere dense" by "residual". The analogous result involving zero measure instead of first category also holds.

T. A. Botts (Charlottesville, Va.).

Erdös, P., and Rado, R. Combinatorial theorems on classifications of subsets of a given set. Proc. London Math. Soc. (3) **2**, 417–439 (1952).

Soit $\Omega_n(x)$ l'ensemble des sous ensembles de n éléments de l'ensemble X. Ce mémoire a pour thème le théorème de Ramsey affirmant l'existence d'une fonction R de trois variables prenant des valeurs entières, telle que, pour tout ensemble S ayant au moins $R(k, n, N)$ éléments et pour toute décomposition de $\Omega_n(S)$ en k classes deux à deux disjointes, il existe au moins un sous ensemble T de S, ayant N éléments et tel que l'ensemble $\Omega_n(T)$ est contenu tout entier dans une certaine classe de cette décomposition, cette propriété n'étant plus vraie lorsque S a moins de $R(k, n, N)$ éléments. Les démonstrations de Ramsey [mêmes Proc. (2) **30**, 264–286 (1929)], de Skolem [Fund. Math. **20**, 254–261 (1933)], de Erdös et Szerkeres [Compositio Math. **2**, 463–470 (1935)], de Erdös et Rado [J. London Math. Soc. **25**, 249–255 (1950); ces Rev. **12**, 322] donnent des estimations majorantes de $R(k, n, N)$ qui sont très larges. L'auteur donne d'abord une modification de ces démonstrations qui conduit à l'estimation suivante, plus serrée que les précédentes:

$$R(k, n, N) \leqq k*k^{n-1}*k^{n-2}*\cdots*k^{2}*(k(N-n)+1)$$

($*$ étant défini par $x*y = x^y$).

Le théorème 2 établit des conditions nécessaires et suffisantes sur n et N pour que toute distribution "invariante" de $\Omega_n(S)$ soit "canonique". (Brièvement, une distribution est dite invariante lorsqu'elle est conservée par toute application de S dans S qui est croissante pour une certaine ordination donnée de S; elle est dite canonique si, étant donné $1 \leqq k \leqq n$ et $1 < \nu_1 < \cdots < \nu_k \leqq n$, une classe de cette distribution est constituée par les sous ensembles de n éléments de S ayant en commun k éléments de rangs $\nu_1, \cdots, \nu_k$ dans la dite ordination.) Le théorème 3 établit alors, grace aux théorèmes 1 et 2, une version finitiste du théorème de Ramsey généralisé au sens Erdös et Rado [loc. cit., p. 251]. Après avoir étudié les extensions transfinies du théorème de Ramsey dans le cas $k = n = 2$, l'auteur montre, par un certain nombre d'exemples, que dans certaines directions le théorème de Ramsey ne peut être généralisé. Certains de ces exemples posent des problèmes non résolus. Un théorème de van der Waerden [Nieuw Arch. Wiskunde (2) **15**, 212–212 (1927)] affirme l'existence d'une fonction W de deux variables, à valeurs entières, telle que, quels que soient les entiers k et l et quelle que soit la décomposition de l'ensemble des entiers positifs inférieurs ou égaux à $W(k, l)$ en k classes disjointes, au moins une de ces classes contient une progression arithmétique de longueur $l+1$, cette propriété n'étant plus vraie pour l'ensemble des entiers strictement inférieurs à $W(k, l)$. Le dernier paragraphe établit l'estimation minorante: $W(k, l) > (2lk)^{1/2}$.

A la page 430, lignes 10 et 13 à partir du bas, ajouter $(b_\mu + b_\nu)(t_\mu - t_\nu) > 0$ pour $\mu < \nu$. *J. Riguet* (Paris).

Rado, R. Direct decomposition of partitions. J. London Math. Soc. **29**, 71–83 (1954).

Soit $A = B_1 \times \cdots \times B_l$ le produit cartésien de l'ensembles donnés. Le problème étudié ici est le suivant: Etant donné une relation d'équivalence R sur A existe t'il un système de "grands" sous ensembles $C_\lambda \subset B_\lambda$ ayant la propriété suivante: la restriction de R à $C_1 \times \cdots \times C_l$ est le produit direct des relations d'équivalences R_λ, projection de R sur les C_λ?

Le théorème 1 établit essentiellement que lorsque B_1 est infini et lorsque $B_2, \cdots, B_l$ sont finis et suffisamment grands, il existe de tels systèmes C_λ, où C_1 est infini et ou C_λ ($\lambda \neq 1$) est un sous ensemble arbitrairement grand de B_λ. La démonstration s'obtient a partir du théorème 1 d'un travail antérieur [Erdös-Rado, 1950, voir référence dans l'analyse ci-dessus] par un processus de polarisation. Le théorème 2 établit dans quelle mesure l'existence de systèmes C_λ peut encore être affirmée lorsque tous les B_λ sont infinis. Le théorème 3 est la version finitiste du théorème 2 et est utilisé dans la démonstration du théorème 1. Sa démonstration fait appel à la version finitiste du théorème de Ramsey généralisé au sens Erdös-Rado mentionné dans l'analyse précédente.

Dans le cas particulier $l = 2$, l'application de ces théorèmes a la relation d'équivalence qui, étant donné une matrice α, a pour classes les différentes "lignes de niveau" de α, permet d'énoncer d'intéressantes propositions sur l'existence de sous matrices de α de certains types. Par exemple, le théorème 2

a pour corollaire la proposition suivante: Toute relation binaire sur un ensemble infini ou sa complémentaire contient une sous relation infinie du type

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ . & . & . & . & \cdots \end{bmatrix} \quad \text{ou du type} \quad \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ . & . & . & \cdots \end{bmatrix}.$$

J. Riguet (Paris).

Kövari, T., Sós, V. T., and Turán, P. On a problem of K. Zarankiewicz. Colloquium Math. 3, 50–57 (1954).

Soit E, F deux ensembles finis ayant respectivement n_1 et n_2 éléments et soient j_1 et j_2 deux entiers, $j_1<n_1$, $j_2<n_2$. Soit k_{j_1,j_2} la fonction de deux variables à valeurs entières définie par l'énoncé suivant: $k_{j_1,j_2}(n_1,n_2)$ est le nombre minimum d'éléments que doit contenir la relation binaire $R\subset E\times F$ pour que l'existence de deux sous ensembles $X\subset E$, $Y\subset F$ ayant respectivement j_1 et j_2 éléments et tels que $X\times Y\subset R$ soit assurée. La fonction d'une variable k_j est définie par $k_j(n)=k_{j,j}(n,n)$.

L'utilisation implicite du lemme suivant "$A_1,\cdots,A_n$ étant n sous ensembles d'un ensemble donné ($\#X$ désignant le nombre d'éléments de X), l'inégalité:

$$(j-1)\#(A_1\cup\cdots\cup A_n)<\#A_1+\cdots+\#A_n$$

entraine l'existence d'indices $i_1,\cdots,i_j$ distincts tels que $A_{i_1}\cap\cdots\cap A_{i_j}\neq 0$" et de l'inégalité de Minkowski permet à l'auteur de démontrer d'une manière simple et élégante que l'on a $k_j(n)<1+jn+[(j-1)^{1/j}n^{(2j-1)/j}]$ (formule (1.5) typographiquement erronée). L'auteur démontre ensuite que, lorsque n tend vers l'infini, $k_2(n)$ est un infiniment grand équivalent à $n^{3/2}$ et suggère qu'il est vraisemblable que pour j quelconque $k_j(n)/n^{(2j-1)/j}$ tend vers une limite finie. Le dernier paragraphe de ce mémoire contient une démonstration de la formule: $k_{2,2}(p^2+p,p^2)=p^2(p+1)+1$ où p est premier. *J. Riguet* (Paris).

Sodnomov, B. S. Example of two sets of type G_δ whose arithmetic sum is non-B-measurable. Dokl. Akad. Nauk SSSR (N.S.) 99, 507–510 (1954). (Russian)

The author proved previously [same Dokl. (N.S.) 80, 173–175 (1951); MR 13, 542] that the direct sum of two B-sets might be a non-B-measurable A-set. In the present paper the same fact is proved directly and the author describes a regular elementary procedure to get two plane G_δ-sets whose direct sum is a non-B-measurable A-set. The sum of two F_σ sets is again an F_σ-set. On the other hand, there exists a non-B-measurable A-set which is not the sum of two B-sets. *G. Kurepa* (Zagreb).

Agnew, Ralph Palmer. Frullani integrals and variants of the Egoroff theorem on essentially uniform convergence. Acad. Serbe Sci. Publ. Inst. Math. 6, 12–16 (1954).

A corollary of Egoroff's theorem. Let $f(A,x)$ be defined for $A>A_0$ and x in a set E with finite positive measure $|E|$. Let $f(A,x)\to f(x)$ as $A\to\infty$, $x\in E$, where $f(x)$ is finite. Let $f_n(x)=\sup|f(A,x)-f(x)|$, $n\leqq A\leqq n+1$, be measurable on E. Then to each $\theta>0$ there corresponds a subset E_1 of E with $|E_1|>|E|-\theta$ such that $f(A,x)\to f(x)$ uniformly over E_1 as $A\to\infty$. An example is given to show that Egoroff's theorem may fail if the functions of the sequence to which the theorem applies are nonmeasurable. There is also given an example which shows that, if the measurability of the functions $f_n(x)$ in the corollary of Egoroff's theorem is replaced by the measurability of $F(A,x)$ and the measurability of F in each variable separately, the corollary fails to hold. *R. L. Jeffery* (Kingston, Ont.).

Ulam, S. M., and Hyers, D. H. On the stability of differential expressions. Math. Mag. 28, 59–64 (1954).

Gli Autori si propongono di esaminare se la conoscenza del comportamento delle derivate di una funzione in un certo punto permette di dire qualcosa circa il comportamento delle derivate delle funzioni prossime a quella data. Per esempio, gli Autori dimostrano che: Se $f(x)$ è definita in un intervallo N col centro nel punto a e vi è derivabile n volte, se $f^{(n)}=0$ ed $f^{(n)}(x)$ cambia segno al passare di x per a, dato $\epsilon>0$ si può trovar $\delta>0$ in guisa, che ogni funzione $g(x)$, soddisfacente in tutto N alla disuguaglianza $|f(x)-g(x)|<\epsilon$ e derivabile n volte, abbia una derivata n-esima che si annulla in qualche punto b, variabile al variare della $g(x)$, soddisfacente alla $|b-a|<\epsilon$. Per $n=1$, gli Autori considerano anche il caso che a cambiar di segno, annullandosi, sia un'espressione del tipo $f'(x)-F(x,f(x))$ e dànno anche un teorema relativo a funzioni di due variabili.

G. Scorza Dragoni (Padova).

Bledsoe, Woodrow W., Norris, Michael J., and Rose, Gene F. On a differential inequality. Proc. Amer. Math. Soc. 5, 934–939 (1954).

The authors consider matrix functions of a real variable and their derivatives. A matrix with elements which are functions of x is said to be non-negative, continuous, essentially bounded, $>-\infty$, etc., if these statements are true of every element in the matrix. If F is a matrix function they write $F=F_1+F_2$, where F_1 is non-negative and F_2 is non-positive. The main theorem is as follows: Let F be an $n\times 1$ matrix function and M an $n\times n$ matrix function. Let (i) F_2 be continuous, $D^+F_2>-\infty$ everywhere, and $F(a)$ be non-negative, (ii) M be essentially bounded above and all its elements not on the principal diagonal be non-negative, (iii) $D^+F\geqq MF$ almost everywhere in (a,b). Then F is non-negative in the interval (a,b). They prove analogous theorems concerning a function Y which satisfies an equation $D^+Y=MY+K$, show that certain of the above conditions on M cannot be completely relaxed, and finally consider more general inequalities in which MF is replaced by a function $G(F,x)$. *U. S. Haslam-Jones* (Oxford).

Pezzana, Mario. Sulla differenziabilità delle funzioni di più variabili reali. Rend. Sem. Mat. Univ. Padova 23, 299–309 (1954).

In a domain D of Euclidean n-space let there be given a real-valued function $f(P)=f(x_1,\cdots,x_n)$. To simplify the statements, we shall assume that f is continuous (the author assumes only continuity with respect to certain groups of the variables). If $n=2$, then the mere existence of the first partial derivatives of f a.e. in D is known to imply the existence a.e. of a so-called regular asymptotic differential. The purpose of the paper is to make a contribution to the fragmentary knowledge available in the general case $n\geqq 2$. Let $\bar P=(\bar x_1,\cdots,\bar x_n)$ be a fixed and $P=(x_1,\cdots,x_n)$ a variable point in D. Let k be one of the integers $1,\cdots,n-1$. Then f is said to admit of an $(n-k)$-asymptotic differential at $\bar P$ if there exist constants $a_1,\cdots,a_n$ and a set $E(\bar P)$ such that (i) the projections of the complement of $E(\bar P)$ upon the $(n-k)$-dimensional coordinate subspaces through $\bar P$ are of $(n-k)$-dimensional Lebesgue density zero at $\bar P$, and (ii) one has

$$\frac{f(P)-f(\bar P)-\sum_{i=1}^n a_i(x_i-\bar x_i)}{P\bar P}\to 0 \quad \text{for} \quad P\in E(\bar P),\ P\to\bar P.$$

The author proves the following theorem: if f admits a.e. of a differential in the sense of Stolz with respect to each group of k of the variables, then f admits a.e. of an $(n-k)$-asymptotic differential. The author notes that for $k=1$ it is sufficient to assume the existence a.e. of the first partial derivatives. *T. Radó* (Columbus, Ohio).

Dubrovskiĭ, V. M. On the method of iterations. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 127–133 (1954). (Russian)

The author is concerned with the possibility of solving the equation (*) $x=f(x)$ by the iterative process (**) $x_n=f(x_{n-1})$, $n=1, 2, \cdots$, where x_0 is arbitrary. (I) Let f be a real-valued continuous monotone increasing function on $[a, b]$ with $f(a)>a$, $f(b)<b$, and let there be a unique solution for (*) in $[a, b]$. Then for any x_0 in $[a, b]$ the sequence $\{x_n\}$ converges monotonically to the solution. If $f(a)<a$, $f(b)>b$, a similar result is obtained by using the inverse function f^{-1} to form the iterations. (II) Suppose f is continuous and decreasing on $[0, a]$, $f(0)\leqq a$, $f(a)\geqq 0$. In order that the iteration (**) converges to (the unique) solution of (*) for an arbitrary choice of x_0, it is necessary and sufficient that the graph of $y=f(x)$ does not have a pair of points which are symmetric with respect to the line $y=x$. He shows that a sufficient condition for the non-existence of such symmetric points is that $d^2f[f(x)]/dx^2\neq 0$ for x in $(0, a)$. Two similar theorems are proved which do not assume monotoneity, but which require knowledge of the solution, so that their applicability is dubious. These results are somewhat more general than some obtained by S. P. Pul'kin (=Poulkine) [Izv. Akad. Nauk SSSR. Ser. Mat. **6**, 71–108 (1942); Dokl. Akad. Nauk SSSR (N.S.) **73**, 1129–1132 (1950); MR **4**, 213; **12**, 395] in that they impose no differentiability restrictions. *R. G. Bartle.*

Kolesova, E. V. On the theory of implicit functions. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 461–476 (1954). (Russian)

The author considers solutions $y=f(x)$ of the equation $F(x, y)=0$ where the function F defines a continuous mapping, into Euclidean p-space with origin 0, of a closed domain of the product space of Euclidean m- and n-spaces in which the variables are x and y. He shows that the set in which there is a solution is an F_σ set E; moreover, that f can be chosen of class 1 in E, and such that its points of continuity are dense in a somewhat stronger sense than usual. He derives this result from a corresponding statement concerning uniformisation of a closed set in the product space. *L. C. Young* (Madison, Wis.).

Darbo, Gabriele. Convergenza in variazione in senso forte e derivazione per serie. Rend. Sem. Mat. Univ. Padova **23**, 310–315 (1954).

Let $\{f_n\}$ be a pointwise convergent sequence of real-valued functions of bounded variation on the real interval $[a, b]$. Where V_a^b denotes total variation on $[a, b]$, the author proves that in order that

$$\lim_{n\to\infty} \overset{b}{\underset{a}{V}}[f_n(x)-f(x)]=0$$

with f an absolutely continuous function on $[a, b]$ it is necessary and sufficient that

$$\lim_{\substack{n\to\infty\\ h\to 0+}} \overset{b-h}{\underset{a}{V}}[f_n(x+h)-f_n(x)]=0.$$

This result constitutes a definitive strengthening of a theorem of E. Baiada [Ann. Scuola Norm. Super. Pisa (3) **6**, 59–68 (1952); MR **14**, 628] as improved by R. Conti [Rend. Sem. Mat. Univ. Padova **23**, 86–90 (1954); MR **15**, 693]. *T. A. Botts* (Charlottesville, Va.).

Gonçalves, Vicente. On the total variation of discontinuous functions. Ciência **4**, nos. 9–10, 9–12 (1954). (Portuguese)

Portuguese-language version of Univ. Lisboa. Rev. Fac. Ci. A. (2) **3**, 137–142 (1954); MR **16**, 120. *T. A. Botts* (Charlottesville, Va.).

Tompson, Robert N. Areas of k-dimensional nonparametric surfaces in $k+1$ space. Trans. Amer. Math. Soc. **77**, 374–407 (1954).

The author studies two types of continuous mappings (surfaces): (a) mappings from a cell C of the k-dimensional Euclidean space E_k into the space E_{k+1} of the type $x_1=u_1, \cdots, x_k=u_k$, $x_{k+1}=f(u_1, \cdots, u_k)$; and (b) mappings from $C\subset E_k$ into E_k of the type $x_1=u_1, \cdots, x_{k-1}=u_{k-1}$, $x_k=f(u_1, \cdots, u_k)$. In both cases f is a continuous single-valued real function of $u=(u_1, \cdots, u_k)$ in the closed cell C. The orthogonal projections of the mappings of the type (a) on the k-dimensional hyperplanes $x_j=0$, $j=1, 2, \cdots, k$, of E_{k+1} are of the type (b). Also, a third type (c) of mappings is considered, namely, the orthogonal projections of the mappings (a) on an arbitrary k-dimensional hyperplane of E_{k+1}. The main following theorems concerning surfaces of the type (a) are proved: (1) the k-dimensional Lebesgue area is finite if and only if the real, continuous, single-valued function f is of bounded variation in the sense of Tonelli (BVT); (2) the same area is $\geqq$ the classical area integral and the equality sign holds if and only if f is absolutely continuous in the sense of Tonelli (ACT); (3) Lebesgue area, integral-geometric area, and other areas are equal; (4) the area derivative is almost everywhere equal to the value of the integrand in the classical area integral; (5) The Lebesgue areas of the smooth surfaces T_n, obtained by replacing f by the integral means f_n of f, approach the Lebesgue area of T as $n\to\infty$. All these results are the natural extensions of previous ones of Tonelli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **3**, 357–362, 445–450, 633–638, 714–719 (1926); **5**, 313–318 (1927)], of T. Radó [Fund. Math. **10**, 197–210 (1927)], and of H. Federer [Trans. Amer. Math. Soc. **55**, 420–437, 438–456 (1944); **59**, 441–466 (1946); MR **6**, 44, 45; **7**, 422] for $k=2$. The notations used are mostly Federer's. First mappings of the type (b) are studied and the results applied to mappings of the type (a). *L. Cesari* (Lafayette, Ind.).

Goffman, Casper. Lower-semi-continuity and area functionals. I. The non-parametric case. Rend. Circ. Mat. Palermo (2) **2** (1953), 203–235 (1954).

The author proposes various extensions of Lebesgue area theory to discontinuous surfaces. He begins by defining the distance between two functions on $[0, 1; 0, 1]$ as

$$\delta(p, q)=\iint |p(x, y)-q(x, y)|\,dxdy,$$

i.e. their distance in L_1. With this as a metric, Lebesgue area is lower semicontinuous (lsc) on the space of polyhedral functions. Fréchet extension yields a functional $\Phi(f)$ lsc on L_1. This functional is the first extension he considers. He proves first that it coincides with Lebesgue area on continu-

ous surfaces. He then introduces modified expressions of Geöcze and proves that the Geöcze area $H(f)$ thus defined coincides with $\Phi(f)$ for all $f \in L_1$. This generalizes a classical result due to Radó [Fund. Math. **10**, 197–210 (1927)] in the continuous case.

This last result is used to prove that $\Phi(f)$ coincides for $f \in L_1$ with the area $\Phi_C(f)$ introduced by Cesari [Ann. Scuola Norm. Sup. Pisa (2) **5**, 299–313 (1936)]. Combining this fact with the result of Cesari [loc. cit.] that $\Phi_C(f) < \infty$ if and only if f is gBVT (of generalized bounded variation in the sense of Tonelli, for which see Cesari, loc. cit.) and, again making use of the Geöcze expressions, he finds that $\Phi(f) < \infty$ if and only if f is gBVT. If f is gBVT, then

$$\Phi(f) \geqq \iint (f_x^2+f_y^2+1)^{1/2} dxdy,$$

the derivatives existing almost everywhere (in a generalized sense). He proves that equality holds if and only if f is gACT, generalized absolutely continuous in the sense of Tonelli, a concept introduced by Calkin [Duke Math. J. **6**, 170–186 (1940); MR **1**, 208] and Morrey [ibid. **6**, 187–215 (1940); MR **1**, 209].

He suggests another extension of Lebesgue area, gotten by defining

$$d(f, g) = \text{measure} \underset{(x,y)}{E} [f(x, y) \neq g(x, y)].$$

Lebesgue area is lsc on the space of continuous functions thus metrized, and thus may be Fréchet-extended to an lsc functional on the completion, i.e. the space of measurable functions. The author has not been able to decide whether this last functional coincides with the previous ones. There is also a treatment of bounded variation for functions of one variable.

J. M. Danskin (Washington, D. C.).

Goffman, Casper. Lower semi-continuity and area functionals. II. The Banach area. Amer. J. Math. **76**, 679–688 (1954).

The author proves by examples that the methods used in the paper reviewed above to extend Lebesgue area to discontinuous nonparametric surfaces do not work in the parametric case. These methods were based on Fréchet's theorem on the extension of a lower semicontinuous nonnegative real function defined on a metric space to the completion of that space. The original space was taken to be the space of polyhedral surfaces. This failure leads the author to ask: is there any functional defined on the space of parametric polyhedral surfaces, lsc with respect to some metric, yielding the elementary area when applied to the polyhedral surfaces, for which the completion of the space contains a reasonably wide class of discontinuous mappings along with all the continuous ones?

He answers this question in the affirmative, but in a sense not as satisfactorily as in the nonparametric case: with his metric, too complicated to reproduce here, the completion is to a space containing all Baire mappings. The extension he gets coincides with the Banach area, rather than with the Lebesgue area, but only for planar mappings. If the mapping is not planar, the Banach area is less than or equal to the extended functional.

J. M. Danskin.

Have

Theory of Functions of Complex Variables

***Stoilow, S. Teoria funcţiilor de o variabilă complexă. Vol. I. Noţiuni şi principii fundamentale. [Theory of functions of a complex variable. Vol. I. Notions and fundamental principles.]** Editura Academiei Republicii Populare Române, Bucharest, 1954. 308 pp. Lei 10.65.

This volume embodies the author's lectures at the University of Bucharest. The first part is a rather condensed introduction to the general theory along fairly conventional lines, beginning with power series. The later chapters take up such topics as entire and meromorphic functions, doubly periodic functions, conformal mapping on the boundary of a Jordan region, multiple-valued functions, and applications of modular functions to the Picard circle of ideas. There are no exercises. *R. P. Boas, Jr.* (Evanston, Ill.).

Borozdin, K. V. On a possible generalization of a theorem of Heilbronn and Landau. Dokl. Akad. Nauk SSSR (N.S.) **98**, 705–707 (1954). (Russian)

Heilbronn and Landau [Math. Z. **37**, 18–21 (1933)] proved that if $c_n \geqq 0$, and $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$, and $f(z) - (1-z)^{-1}$ is regular at $z=1$, then $c_n = O(1)$. The author proves that if $a_n \geqq -\gamma n^\delta$ $(\gamma > 0, 0 \leqq \delta < 1)$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 1$, and near $z=1$

$$f(re^{i\theta}) = O(|\theta|^{-\alpha}) \quad \text{for} \quad r \leqq 1 - |\theta|^\beta,$$

where $\beta \geqq 1$ and $0 \leqq \alpha < 1$, then $a_n < \Gamma n^{\delta+1/\beta}$, where Γ is a constant. The proof is based on the same principle: use of the Féjer kernel. *H. Davenport* (London).

Rosen, David. A class of continued fractions associated with certain properly discontinuous groups. Duke Math. J. **21**, 549–563 (1954).

The author wants to characterize arithmetically the coefficients of the linear fractional transformations

$$z' = V(z) = (az+b)/(cz+d), \quad ad-bc=1$$

in the group $\Gamma(\lambda)$ generated by

$$S = S(z) = z+\lambda, \quad T = T(z) = -1/z, \quad I = I(z) = z.$$

According to Hecke this group is properly discontinuous only if $\lambda = 2\cos \pi/q$, q integer $\geqq 3$, when $\lambda < 2$ and for every real λ when $\lambda > 2$. If $V(z) = S^{r_0}TS^{r_1}T\cdots TS^{r_n}(z)$ is expressed as a "word" in S and T, the author associates to $V(z)$ the continued fraction ("λ-fraction")

$$(r_0\lambda, -1/r_1\lambda, -1/r_2\lambda, \cdots, -1/r_n\lambda).$$

By means of the relation $(TS)^2 = I$, a "word" can be transformed into a reduced form of minimal length and two words define the same substitution if and only if their reduced forms are identical (the case $q=3$, $\lambda=1$ is excluded from the discussion). The author further discusses the arithmetic properties of infinite reduced λ-fractions: The denominators of the convergents are non-decreasing and tend to ∞; every infinite reduced λ-fraction converges and for a given λ ($\lambda = 2\cos \pi/q$, $q \geqq 4$ or $\lambda > 2$) a real number is represented by a reduced λ-fraction in only one way. The author finally gives an algorithm by which any real number can be developed into a reduced λ-fraction and he shows that the approximation by its convergents is of the same order of accuracy as in the case of the regular continued fraction expansions. *H. D. Kloosterman* (Leiden).

Have

★**Picone, Mauro. Sul calcolo delle funzioni olomorfe di una variabile complessa.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 118–126. Academic Press Inc., New York, 1954. $9.00.

The author develops formulas, similar to the Cauchy integral formula, which show how a knowledge of the values of a function $f(z)$, analytic and bounded in a domain whose frontier consists of a finite number of regular curves, on a portion of the frontier can determine completely $f(z)$ and its derivatives in the domain. It is remarked that analogous results have been obtained by G. Zin [Ann. Mat. Pura Appl. (4) **34**, 365–405 (1953); MR **14**, 1073].

A. J. Lohwater (Ann Arbor, Mich.).

Gel'fer, S. A. The variation of multivalent functions. Dokl. Akad. Nauk SSSR (N.S.) **98**, 885–888 (1954). (Russian)

The author considers the class of functions $z=f(\zeta)$ which are (1) regular in $|\zeta|<1$, with $f(0)=0$, (2) map $|\zeta|<1$ onto a region D lying in some p-sheeted algebraic Riemann surface R of genus γ, (3) omit $a_1, a_2, \cdots, a_m$ in $|\zeta|<1$. To obtain a variational formula for functions in this class, the universal covering surface S of R is considered. Let $\Phi(Z)$ map (a) the extended plane, (b) the plane, (c) the unit circle onto S (for each R exactly one of these cases occurs). Then $Z=\Phi^{-1}(f(\zeta))$ is univalent in the unit circle and known variational formulas can be applied. If $Z^*(\zeta)$ is a variation of $Z(\zeta)$, then $f^*(\zeta)=\Phi(Z^*(\zeta))$ gives the desired variation of $f(\zeta)$. The resulting formulas are complicated and involve $\Phi(Z)$ explicitly. It seems to the reviewer that in the last two terms of formulas (7) and (10), h should be replaced by $|h|$.

A. W. Goodman (Lexington, Ky.).

Gel'fer, S. A. On typically real functions of order p. Mat. Sb. N.S. **35**(77), 193–214 (1954). (Russian)

Let $T(p)$ denote the class of functions $\sum_{n=1}^{\infty} a_n z^n$ which are regular in $|z|<1$, have all coefficients real, and for which there is a $\rho<1$ such that $\Im f(z)$ changes sign $2p$ times on each circle $|z|=r$, $\rho<r<1$. The reviewer and Robertson [Trans. Amer. Math. Soc. **70**, 127–136 (1951); MR **12**, 691] proved that if $f(z) \in T(p)$ then for each $n>p$

$$(1)\quad |a_n| \leq \sum_{k=1}^{p} |a_k| \frac{2k(n+p)!}{(n^2-k^2)(p+k)!(p-k)!(n-p-1)!}$$

and this inequality is sharp in all of the variables.

The author now gives a second proof of (1) which is somewhat longer than the original. It is proved that if $f(z) \in T(p)$, then it can be represented in the form

$$(2)\quad f(z)=\sum_{m=1}^{p-1} A^{(m)} \prod_{k=1}^{m} s(z, \cos\theta_k) + \frac{1}{\pi}\prod_{k=1}^{p-1} s(z,\cos\theta_k)\int_0^{\pi} s(z,\cos\theta)d\mu(\theta),$$

where $A^{(1)}, \cdots, A^{(p-1)}$ are real, $\mu(\theta)$ is a real non-decreasing function and $s(z,x)=z/(1-2zx+z^2)$. The class of functions given by the right side of (2) is wider than $T(p)$, so the bound (1) is now extended to this wider class.

A. W. Goodman (Lexington, Ky.).

Zmorovič, V. A. On some special classes of analytic functions univalent in a circle. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 175–182 (1954). (Russian)

Let $f(z)=z+\cdots$ be regular and univalent in $|z|<1$ and map $|z|=1$ onto a closed smooth convex curve. (XI) If $0<a<\frac{1}{2}$, $1+af''(0)\neq 0$, $\beta=(1-\alpha)/(1+af''(0))$, $\alpha\neq 0$, and $|\arg\alpha-\arg\beta|<\pi/2$, then

$$F(z)=\alpha f(z)+\beta f'(z)(a+z+az^2)-a\beta$$

is univalent in $|z|<1$. (XII) If $0<\alpha<2\pi$, $0<a<\frac{1}{2}$, and $b=1+a\cos\alpha f''(0)\neq 0$, then

$$F(z)=\frac{1}{b}\left[\frac{f(ze^{i\alpha})-f(ze^{-i\alpha})}{e^{i\alpha}-e^{-i\alpha}}\cdot\frac{a+z+az^2}{z}-a\right]$$

is univalent and starlike in $|z|<1$. The other ten theorems mentioned in this paper are all well known.

A. W. Goodman (Lexington, Ky.).

Kufarëv, P. P., and Semuhina, N. V. On a problem of N. N. Luzin. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 183–185 (1954). (Russian)

Luzin [Doklady Akad. Nauk SSSR (N.S.) **56**, 447–450 (1947); MR **9**, 181] has raised the question: does there exist a function holomorphic and bounded in $|z|<1$ such that for almost all points ζ on the boundary Γ, $|z|=1$, each circle tangent to Γ at ζ on the inside, is mapped into a region of infinite area. In this direction the authors prove the existence of a function for which there is a set of such points everywhere dense on Γ. *A. W. Goodman* (Lexington, Ky.).

Combes, Jean. Sur la détermination des fonctions analytiques par des conditions imposées à leurs dérivées successives. Bull. Sci. Math. (2) **78**, 199–216 (1954).

The principal results were announced in C. R. Acad. Sci. Paris **237**, 1482–1484 (1953); MR **15**, 412. In an appendix the author rediscovers a theorem of Ålander's on regions in which an infinity of derivatives of an entire function are zero-free [cf. Pólya, Bull. Amer. Math. Soc. **49**, 178–191 (1943), in particular p. 181; MR **4**, 192].

R. P. Boas, Jr. (Evanston, Ill.).

Shah, S. M., and Singh, S. K. On the maximum function of a meromorphic function. Math. Student **22**, 121–128 (1954).

The authors construct meromorphic functions negating three theorems of S. K. Bose [Math. Z. **56**, 223–226 (1952); MR **15**, 23]. *R. P. Boas, Jr.* (Evanston, Ill.).

Montel, Paul. Sur un critère principal de normalité. J. Analyse Math. **3**, 209–224 (1954).

For a family F of functions f of the complex variable z, meromorphic in a domain D, a condition that is both necessary and sufficient for the normality of the family is said to be a principal criterion; a condition that is sufficient but not necessary is called secondary. Thus the criterion of Marty relative to the spherical derivative is principal, while the criterion of three exceptional values is secondary.

The present paper is devoted to a proof and some applications of the following principal criterion: In order that a family F of functions $f(z)$, meromorphic in a domain D, be normal in the interior of D, it is necessary and sufficient that, in every domain D' interior to D, the roots of the equations $f(z)=a$, $f(z)=b$, $f(z)=c$, $f(z)=d$, where a, b, c, d are fixed and distinct complex numbers, be such that the distance of each root of any one of these equations from each root of any of the others is greater that a fixed positive ϵ independent of the member of the family

E. F. Beckenbach (Los Angeles, Calif.).

Jenkins, James A. Some uniqueness results in the theory of symmetrization. Ann. of Math. (2) **61**, 106–115 (1955).

The author shows that if one applies circular symmetrization to a doubly connected domain then its modulus is strictly decreased unless the symmetrization amounts to a rotation about the origin. This result is applied to give similar results for the modulus of a quadrilateral and the inner radius of a simply connected domain. These results enable one to discuss the unicity of extremals to problems which have been attacked by circular symmetrization.

H. L. Royden (Stanford, Calif.).

Hiong, King-Lai. Généralisations du théorème fondamental de Nevanlinna-Milloux. Bull. Sci. Math. (2) **78**, 181–198 (1954).

The author establishes inequalities of which the following is typical

$$(*)\quad T(r,f)<lN(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f_l-\psi}\right)+S(r).$$

Here $f(z)$ is meromorphic,

$$f_l(z)=\sum_{i=0}^{l}\alpha_i(z)f^{(i)}(z),$$

the functions $\alpha_i(z)$, $\psi(z)$ have smaller growth than $f^{(i)}(z)$, $f(z)$, respectively, and the notation is that of Nevanlinna theory. The proofs are similar to those of the special cases when the $\alpha_i(z)$ are nonzero, and $\psi(z)\equiv 1$, due to Milloux [Les fonctions méromorphes et leurs dérivées, Hermann, Paris, 1940; MR 7, 427]. Further, the coefficient l of $N(r,f)$ in (*) may be replaced by 1, as was done in a similar result of the reviewer [Rend. Circ. Mat. Palermo (2) **2**, 346–392 (1954); MR **16**, 122]. *W. K. Hayman* (Exeter).

Collingwood, E. F. On the linear and angular cluster sets of functions meromorphic in the unit circle. Acta Math. **91**, 165–185 (1954).

The present paper gives proofs and elaboration of results announced in two earlier notes [Proc. Internat. Congr. Math., Amsterdam, Sept. 1954, vol. II, p. 91, Noordhoff, Amsterdam, 1954; C. R. Acad. Sci. Paris **238**, 1769–1771 (1954); MR **15**, 863]. Let $f(z)$ be meromorphic in $|z|<1$. The sets $S(f)$, $I(f)$, and $F(f)$ are defined, respectively, as the set of points $z=e^{i\theta}$ for each of which the corresponding radial cluster set of $f(z)$ is the whole complex plane, the set of Plessner points of $f(z)$, and the set of Fatou points of $f(z)$ on $|z|=1$. Relations between these three sets are studied in detail. We cite, as typical, the result that $S(f)$ and $I(f)$ differ by a set of first category on $|z|=1$. As pointed out by the author, some of the theorems overlap with results obtained by Bagemihl and Seidel [Proc. Nat. Acad. Sci. U. S. A. **39**, 1068–1075 (1953); MR **15**, 295].

W. Seidel (South Bend, Ind.).

Bagemihl, F., and Seidel, W. Some boundary properties of analytic functions. Math. Z. **61**, 186–199 (1954).

The paper deals with cluster sets of functions that are analytic for $|z|<R$, where $0<R\leqq+\infty$. Some of its theorems and corollaries are generalizations of certain results due to A. Roth [Comment. Math. Helv. **11**, 77–125 (1938)]; and the proof of the first principal theorem of the paper is largely based on a recent approximation theorem by S. N. Mergelyan [Dokl. Akad. Nauk SSSR (N.S.) **78**, 405–408 (1951); MR **13**, 23; **14**, 858].

The authors define a monotonic boundary path in $|z|<R$ as a continuous curve $z=z(t)$, $0\leqq t<1$, where $|z(t)|$ increases monotonically from 0 to $R-0$ as t goes from 0 to $1-0$. They then introduce the notion of a tress in $|z|<R$; the simplest example of such a tress is a family of radii of $|z|<R$ whose end points on $|z|=R$ form a set of first category and of type F_σ. (Obvious changes in language are necessary here and in what follows when $R=+\infty$.) The general idea of a tress can be roughly described as a generalization of the example above, obtained by replacing the radii by more general monotonic boundary paths. The principal results of the paper will be briefly described below.

To every continuous function $\phi(z)$ in $|z|<R$, an analytic function $f(z)$ in $|z|<R$ can be found such that, on each monotonic boundary path of a given tress, the functions f and ϕ (also $\Re f$ and $\Re\phi$ as well as $\Im f$ and $\Im\phi$) have the same cluster sets. This theorem can be used to construct functions $f(z)$, analytic in $|z|<R$, which show certain remarkable behaviors as $|z|\to R$. For instance, $f(z)$ may approach a given value c (finite or ∞) on more than denumerably many spirals with infinitely many windings each; or $f(z)$ may approach a given value c from a prescribed direction as z approaches the circle $|z|=R$ along almost every radius of $|z|<R$; or $w=f(z)$ may have the entire unit circle $|w|=1$ as its cluster set on almost every radius of $|z|<R$.

If M is a non-empty, nowhere dense set of radii of $|z|<R$ then an analytic function $w=f(z)$ in $|z|<R$ can be found which is uniformly bounded on M but has no limit on any radius belonging to the closure of M and has the whole w-plane as its cluster set on every other radius. It is possible to construct a function $f(z)$, analytic in $|z|<R$, which has given continua $K_1, K_2, \cdots$ as its cluster sets on a given set of monotonic boundary paths $p_1, p_2, \cdots$, respectively. It is also possible to construct such a function which has a given continuum K as its cluster set on almost every radius of $|z|<R$. In all the above-mentioned cases, whenever $f(z)\equiv c$ would constitute a trivial solution of the problem, it is understood that a non-constant function with the desired properties can be found.

Finally, the notion of the radial limit set of $f(z)$, analytic for $|z|<R$, is introduced: this is the set of all w (finite or ∞) which are the radial limit of $f(z)$ along at least one radius of $|z|<R$. A point-set A in the extended w-plane is a radial limit set if and only if A is an analytic set. *F. Herzog.*

Heins, Maurice. A universal Blaschke product. Arch. Math. **6**, 41–44 (1954).

Verf. fügt den allgemeinen Resultaten über Approximation analytischer Funktionen von Seidel und Walsh [Bull. Amer. Math. Soc. **47**, 916–120 (1941); MR **4**, 10] folgende Präzisierung im Falle beschränkter Funktionen hinzu: Es gibt ein konvergentes Blaschke-Produkt $b(z)$, dessen Nullstellen sich nur in $z=1$ häufen, und eine monoton wachsende Folge $\{x_n\}_0^\infty$ mit $x_0=0$ und $\lim x_n=1$ von der folgenden Art: Zu jeder in $|z|<1$ analytischen Funktion f vom Betrag $\leqq 1$ existiert eine Teilfolge $\{x_{n(k)}\}_0^\infty$, sodass $b([z+x_{n(k)}]/[1+z\cdot x_{n(k)}])$ in $|z|<1$ lokal gleichmässig gegen f konvergiert. *A. Pfluger* (Zürich).

Lohin, I. F. On an interpolation problem for entire functions. Mat. Sb. N.S. **35**(77), 223–230 (1954). (Russian)

The author generalizes the Laplace-Borel transform of an analytic function by transforming $\sum c_n z^n$ into $\sum a_n c_n z^n$, where $\sum a_n z^n$ is regular in a sufficiently large circle and no $a_n=0$. He obtains the integral form of the transform and

discusses interpolation, uniqueness and expansion theorems generalizing those which are familiar in the classical case $a_n=1/n!$. *R. P. Boas, Jr.* (Evanston, Ill.).

Lohin, I. F. On completeness of the system of functions $\{f(\lambda_n z)\}$. Mat. Sb. N.S. **35**(77), 215–222 (1954). (Russian)

By means of Carleman's formula for functions regular in a half-plane the author finds conditions under which some sets of analytic functions $\{f(\lambda_n z)\}$ are complete. An example is the following theorem. Let $E(z)=\sum z^n/\Gamma((n/p)+1)$ $(p>\frac{1}{2})$. If

$$(1)\qquad |\arg \lambda_n|=|\theta_n|<\pi/2p,$$

$$(2)\qquad \limsup_{R\to\infty}\,(1/\log R)\sum_{|\lambda_n|<R}|\lambda_n|^{-p}\cos\theta_n p\geqq p\gamma,$$

then $\{E(\lambda_n z)\}$ is complete in the domain containing $z=0$ and bounded by $|z|^p=\pi\gamma/|\sin p\phi|$; $|\phi|<\pi/p$, $\varphi=\arg z$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Evgrafov, M. A. Completeness of neighboring systems. Dokl. Akad. Nauk SSSR (N.S.) **98**, 525–526 (1954). (Russian)

The author announces several theorems on the completeness (for expanding analytic functions in a region) of a system of functions which is "close" to a given complete system, and on the transformation of a complete system into a complete system by linear operators. *R. P. Boas, Jr.* (Evanston, Ill.).

Evgrafov, M. A. On a recurrence relation connected with the Abel-Gončarov interpolation problem. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 449–460 (1954). (Russian)

Let $\Phi(z)=\sum\alpha_n z^n$ $(\alpha_n\neq 0)$ be an entire function, $l_n(F)$ an additive functional defined for entire functions,

$$\phi_n(\zeta)=l_n(\Phi(z\zeta));$$

$P_m(z)$ the polynomial of degree m such that $l_n(P_m)=\delta_{nm}$. It is assumed that a) $\phi_n(\zeta)=\zeta^n+$higher powers,

b) $$\limsup_{n\to\infty}|\phi_n(z)|^{1/n}\leqq u(z)$$

and except on a set of measure zero $\lim_{n\to\infty}|\phi_n(z)|^{1/n}$ exists and equals $u(z)$. c) The curves $u(z)=\rho$ $(\rho<\rho_0)$ encircle $z=0$ and expand with increasing ρ. d) The formal expansions $z^k=\sum_{n=0}^{\infty}a_{kn}\phi_n(z)$ converge for $u(z)<\rho_1\leqq\rho_0$.

e) $$\sum_{n=k}^{\infty}|a_{kn}|(u(z))^n<(G(z))^k.$$

The author studies the convergence of the interpolation series (1) $\sum_{n=0}^{\infty}l_n(F(z))P_n(\zeta)$ to $F(\zeta)$. Th. 1. If $F(z)=\sum c_n z^n$ is an entire function such that $f(z)=\sum(c_n/\alpha_n)z^{-n-1}$ is regular outside and on $u(z)=\rho_1$, then (1) converges to $F(\zeta)$. Th. 2. The conclusion of Theorem 1 is false for some $F(z)$ for which $f(z)$ is regular in a smaller region only. (It is assumed that ρ_1 is chosen as large as possible).

The second part of the paper deals with the special case $l_n(F)=F^{(n)}(\lambda_n)$. By an ingenious direct discussion of the system of equations determining the a_{kn} in terms of the α_n and the λ, bounds on the a_{kn} are obtained which allow the application of Theorems 1 and 2. Typical conclusions are as follows: If $\lambda_{n+1}-\lambda_n\to 1$, then (2) $\sum F^{(n)}(\lambda_n)P_n(\zeta)$ converges to $F(\zeta)$, if $f(\zeta)=\sum n!c_n\zeta^{-n-1}$ is regular outside $|\zeta e^{\zeta}|<1/e$. If $(\lambda_{n+1}-\lambda_n)/2n\to 1$, then (2) converges to $F(\zeta)$, if $\sum(2n)!c_n\zeta^{-n-1}$ is regular outside

$$|\{(1+\zeta)^{1/2}-1\}\exp\{(1+\zeta)^{1/2}\}|<1.$$

This improves previous results of the author [same Izv. **17**, 421–460 (1953); **18**, 201–206 (1954); MR **15**, 515; **16**, 25]. *W. H. J. Fuchs* (Ithaca, N. Y.).

Hartman, Philip, and Wintner, Aurel. On conformal maps defined by Lagrange's series and by solutions of $dw/dz=f(w)$. Rend. Circ. Mat. Palermo (2) **3**, 282–292 (1954).

The equation $w=zf(w)$, where $f(w)$ is regular in $|w|<1$ with $|f(w)|<1$ and $f(0)\neq 0$, is suggested by Kepler's equation and defines $w(z)\neq 0$ analytic near $z=0$. Classical results indicate that $w(z)$ is regular in $|z|<1$ and that the range D of $w(z)$ lies in $|w|<1$. The authors show that D lies also in the interior of the "Borel polygon" of $1/f(w)$. For the same class of functions $f(w)$, the authors also consider the solution $w(z)$ of $dw/dz=f(w)$ and discuss the range of $w(z)$. *L. Markus* (New Haven, Conn.).

Meschkowski, Herbert. Die Koeffizienten des Bergmanschen Orthonormalsystems. Math. Ann. **128**, 200–203 (1954).

Let B be a finite domain in the complex z-plane whose boundary C consists of n analytic curves C_ν. In a previous paper [Math. Ann. **27**, 107–129 (1954); MR **16**, 348] the author introduced a set of domain functions $N_n(z,u)$ and $M_n(z,u)$ which are regular analytic as functions of z in $B+C$ except for the principal part $(n-1)!/(z-u)^n$ of $N_n(z,u)$ at the parameter point u and which satisfy on C the relations $M_n'dz=\overline{(N_n'dz)}$. He considered further an orthonormal system of analytic functions $\rho_n(z)$ in B which satisfy at u the conditions: $\rho_n(u)=\rho_n'(u)=\cdots=\rho_n^{(n-1)}(u)=0$, $\rho_n^{(n)}(u)=1$ and are characterized by certain extremum properties. In the present paper the author develops $\rho_n(z)=\sum_{\nu=1}^{n}\beta_\nu^{(n)}M_\nu'(z,u)$, where the $\beta_\nu^{(n)}$ are a matrix of numbers characteristic for the domain. Then, each analytic function $f(z)$ with a finite Dirichlet integral over B can be developed into a Fourier series $f(z)=\sum_{n=1}^{\infty}c_n\rho_n(z)$ and the coefficient c_n can be expressed as finite linear combinations of derivatives $f^{(\nu)}(u)$ by means of the matrix $\beta_\nu^{(n)}$. No integration process is, therefore, necessary for the Fourier development, once the $\beta_\nu^{(n)}$ have been obtained. Generalizations to the case of singular functions $f(z)$ and different types of orthonormal systems are indicated. *M. Schiffer.*

Fabian, William. The Riemann surfaces of a function and its fractional integral. Edinburgh Math. Notes no. **39**, 14–16 (1954).

The fractional integrals considered here have the form

$$\frac{1}{\Gamma(\lambda+\gamma)}\left(\frac{d}{dz}\right)^{\gamma}\int_a^z(z-t)^{\lambda+\gamma-1}f(t)\,dt,$$

where the integration is along a simple arc, and ν is the least nonnegative integer such that $\Re(\lambda)+\gamma>0$. The author discusses the character of the branch points of the fractional integral at a point where the behavior of $f(z)$ is specified in various ways. *R. P. Boas, Jr.* (Evanston, Ill.).

Have

✱Schiffer, Menahem, and Spencer, Donald C. Functionals of finite Riemann surfaces. Princeton University Press, Princeton, N. J., 1954. x+451 pp. \$8.00.

Prior investigations of the authors and their collaborators have been concerned with aspects of the theory of the conformal mapping of plane regions in which domain functions, variational methods, and the problem of coefficient domains are central considerations. The object of the present mono-

graph is to give a systematic account of functionals of finite Riemann surfaces and to apply the methods developed to investigations which include as special cases some of the earlier work of the authors. By a finite Riemann surface is meant a compact surface with boundary (consisting of a finite number ($\geqq 0$) of contours) which is endowed with a conformal structure. The classical condition of orientability is not imposed. The orientable double of a finite Riemann surface is employed systematically. A central theme is the dependence of abelian differentials on the surface. In the interests of completeness an introductory account of the theory of Riemann surfaces is given in the first three chapters.

The first chapter starts with an account of the historical background of the modern theory of Riemann surfaces. The remainder of the chapter treats exterior differential forms, the elementary topology of surfaces and integration formulas. The theme of existence theorems for harmonic functions dominates the second chapter. Here the method of orthogonal projection comes to the fore. The existence of harmonic differentials with assigned periods and harmonic functions with assigned singularities and boundary value problems are treated. The chapter culminates in the mapping theorem for planar finite surfaces. The third chapter is concerned with relations between differentials, periods of differentials and the Riemann-Roch theorem. Bilinear differentials and reproducing kernels are studied in the fourth chapter. One objective is to represent the various domain functionals in terms of the Green's function. Thanks to such a representation the problem of the dependence of a functional on the surface is reduced to the special problem of the dependence of the Green's function on the surface. Relations (identities and inequalities) between domain functionals of a given finite Riemann surface and corresponding functionals of an embedding finite Riemann surface are determined in the fifth chapter. With their aid a necessary and sufficient condition is given that a local complex analytic embedding of one finite Riemann surface into another define an analytic embedding in the large of the first surface into the second. The results are specialized to the case of schlicht functions (coefficient problem for schlicht functions in multiply-connected regions, bounded schlicht functions). The sixth chapter treats from a Hilbert-space point of view certain integral operators which map analytic differentials on a subdomain of a finite Riemann surface into piecewise analytic differentials on the surface. The theory is applied to the problem of the representation of domain functionals for a subdomain in terms of the domain functionals of the embedding surface.

The seventh chapter develops the variational calculus for domain functions. Special important types of surface variations are introduced and the associated variations of the domain functionals are determined. The concept of the variational kernel which permits the construction of variations plays a basic role. A necessary and sufficient condition for conformal equivalence under deformation is given. Application is made in the eighth chapter of the variational method to the study of the dependence of the abelian differentials on the moduli and to the coefficient problem for univalent conformal maps of one finite Riemann surface into another. The mappings corresponding to the boundary points of the nth coefficient body are characterized by certain equations involving quadratic differentials. The specialization to plane domains is made and extremal problems for schlicht maps of such domains are treated by the developed methods.

The ninth chapter gives an account of higher dimensional Kähler manifolds. The problem of the generalization of the theory developed in the preceding chapters to Kähler manifolds is discussed. *M. Heins* (Providence, R. I.).

Bergman, Stefan. On zero and pole surfaces of functions of two complex variables. Trans. Amer. Math. Soc. **77**, 413–454 (1954).

In several previous papers the author has introduced (in the space of two complex variables) "domains with a distinguished boundary surface" (which are closely related to the "analytic polyhedra" considered by A. Weil, K. Oka, H. Behnke and others) and "functions of the extended class". The "extended class" is an extension of the class of the real parts of holomorphic functions of two complex variables such that boundary-value problems for domains with a distinguished boundary surface become uniquely solvable. The author has derived integral formulas of "Green's type", i.e. formulas expressing a function of the extended class through its values on the distinguished boundary surface. Using these formulas, the author has obtained functionals which can be considered as a generalization of notions such as "number of points a_ν for which $f(a_\nu)=a$ in the circle $|z|<r$, $\sum_{|a_\nu|\leq r}|a_\nu|^{-\lambda}$, etc." Because zeros and poles form two-dimensional surfaces, the variety of functionals is greater than in one variable. In the present paper functionals of a new type are introduced, in particular functionals associated with a pair of meromorphic functions f_1, f_2. Let $P_1(z_2)$ be the product of distances (measured in a certain metric) of the zeros of f_1 to the zeros of f_2 for fixed z_2, $P_2(z_2)$ the product of the distances of the zeros of f_1 to the poles of f_2, etc. Intersections of zero and pole surfaces of f_1 and f_2 are "cut out" as "exceptional points". The average of $P_1(z_2)P_4(z_2)/P_2(z_2)P_3(z_2)$ (integrated over a certain projection of the four-dimensional domain considered) is interpreted as "generalized Blaschke product". Estimates for this expression are obtained.

In previous papers the functions of the extended class were defined only for continuous boundary values. In the present paper boundary values are admitted which may be infinite at finitely many points. This is necessary because a characteristic method of the paper is, to prescribe the values of $\log|f|$ (f meromorphic) on the distinguished boundary surface as boundary values for a function of the extended class (continuous in the interior of the domain). The values prescribed by $\log|f|$ can be infinite because the zeros and poles of f can intersect the distinguished boundary surface. Some of the integrals considered are complicated, also many little "smoothness" hypotheses are introduced throughout the paper. Finally it is shown that the kernel function for the functions of the extended class with respect to the scalar product $\mathrm{Re}\,[\int(\partial F_1/\partial z_1)(\partial F_2/\partial \bar{z}_1)d\omega]$ (integrated over the whole four-dimensional domain) exists and is finite.

H. J. Bremermann (Münster).

Rothstein, Wolfgang. Der Satz von Casorati-Weierstrass und ein Satz von Thullen. Arch. Math. **5**, 338–343 (1954).

The theorem of Casorati-Weierstrass states that if $g(w)$ is single-valued and regular in $0<|w|<1$ and if it has an essential singularity at the origin, then, for given numbers A, $\epsilon>0$, $\delta>0$, there exists a point a in $0<|w|<\epsilon$ such that $|g(a)-A|<\delta$. This result has been carried over to analytic surfaces by P. Thullen [Math. Ann. **111**, 137–157 (1935)] and others, and a special case of Thullen's result asserts that if $\mathfrak{g}$ is an analytic surface in R: $(|z|<1; 0<|w|<1)$

and if $(0, 0)$ is an essential boundary point of $\mathfrak{g}$ then all points of the circle ($|z|<1$; $w=0$) are essential boundary points of $\mathfrak{g}$.

The author points out that for Thullen's theorem the requirement that $\mathfrak{g}$ be analytic in the entire domain R can be weakened. Namely, if $\mathfrak{g}$ is assumed to be analytic only in $R-N$, where N is closed and of absolute harmonic measure zero, and if $(0, 0)$ is an essential boundary point of $\mathfrak{g}$, then the same conclusion is still valid. The author has shown earlier [ibid. **126**, 221–238 (1953); MR **15**, 616] that, with these same hypotheses on $\mathfrak{g}$, if $\mathfrak{P}$ is any closed set of positive harmonic measure in $|z|<1$, the surface $\mathfrak{g}$ meets at least one place $z=c\in\mathfrak{P}$ infinitely often. This is of course a generalization of the well known Picard theorem on isolated essential singularities.

Generalizations of Thullen's theorem to several dimension have been given by Remmert and Stein [ibid. **126**, 263–306 (1953); MR **15**, 615] and by the author under weakened hypotheses analogous to those just mentioned. This paper is mainly concerned with the proof of a theorem of the Picard type for higher dimensional surfaces: Let $\mathfrak{Z}=\bigcap_1^k(|z_i|<1)\cap\bigcap_1^{n-k}(|w_i|<1)$ and let $N_1, \cdots, N_{n-k}$ be closed sets of absolute harmonic measure zero in the unit circle. Denote by N the union of all k-planes $E^k(c)$:

$$(w_1, \cdots, w_{n-k})=(c_1, \cdots, c_{n-k})\in(N_1, \cdots, N_{n-k}).$$

Further let $M_1, \cdots, M_k$ be closed sets of positive harmonic measure in the unit circle, and let $\mathfrak{P}$ be the set of all $(n-k)$-planes $(z_1, \cdots, z_k)=(c_1, \cdots, c_k)\in(M_1, \cdots, M_k)$. If now $\mathfrak{g}^k$ is an analytic surface of complex dimension k in $\mathfrak{Z}-N$, and if each plane in $\mathfrak{P}$ meets $\mathfrak{g}^k$ in only finitely many points, then the closure $|\mathfrak{g}^k|$ of $\mathfrak{g}$ is an analytic surface in $\mathfrak{Z}$.

By the method of projection the general case is reduced to the case of an analytic hypersurface of complex dimension $n-1$ in n-dimensional space. The author then reduces this case by induction to $n=2$ (contained in results stated above). It is shown by the induction that the closure of $\mathfrak{g}^{n-1}$ in a smaller polycylinder

$$(z_1\in K_1,\ |z_2|<r,\ \cdots,\ |z_{n-1}|<r,\ |w|<r)$$

is an analytic surface. But then if $\mathfrak{g}^{n-1}$ had an essential boundary point $(\zeta_1, \cdots, \zeta_{n-1}, a)$ in $\mathfrak{Z}$, it would follow from an earlier result of the author (loc. cit.) that every point of ($|z_1|<1, \cdots, |z_{n-1}|<1, w=a$) is an essential boundary point of $\mathfrak{g}^{n-1}$, contradicting the result just stated.

J. H. Sampson (Cambridge, Mass.).

Magnus, Arne. Volume-preserving transformations in several complex variables. Proc. Amer. Math. Soc. **5**, 256–266 (1954).

Verf. betrachtet holomorphe und volumentreue Abbildungen von Gebieten des C^2 in den C^2, welche durch die Funktionen $u(z_1, z_2)$, $v(z_1, z_2)$ beschrieben werden; ferner wird noch die Eineindeutigkeit der Abbildungen verlangt. Sind ausserdem

$$u(z_1, z_2)=\sum_{i=1}^{m} f_i(z_1, z_2),\quad v(z_1, z_2)=\sum_{i=1}^{n} g_i(z_1, z_2)$$

mit $f_i(z_1, z_2)$, $g_i(z_1, z_2)$ als homogenen Polynomen vom Grade i, so lassen sich die $f_i(z_1, z_2)$ als Polynome der $g_i(z_1, z_2)$ darstellen und es bestehen darüber hinaus noch gewisse Relationen zwischen den $g_i(z_1, z_2)$. Ist $n\leqq m$ und $n=2$ oder $=3$, so gilt weiter $n|m$; es erfolgt eine ausführliche Behandlung dieser beiden Fälle. Abschliessend werden noch einige Beispiele von eindeutigen, holomorphen und volumentreuen Abbildungen gegeben, bei denen $u(z_1, z_2)$, $v(z_1, z_2)$ keine Polynome mehr sind.

H. Röhrl (München).

Stoll, Wilhelm. Einige Bemerkungen zur Fortsetzbarkeit analytischer Mengen. Math. Z. **60**, 287–304 (1954).

Les résultats nouveaux du mémoire sont, essentiellement: a) un critère métrique pour qu'une variété complexe W^{n-1} dans l'espace C^n, définie dans $D-W^q$ (W^q est une variété analytique complexe de dimension complexe $q\leqq n-1$, D un domaine de C^n) soit prolongeable dans D: il faut et il suffit qu'il existe (avec des notations un peu différentes de celles de l'auteur) une forme $\alpha=\alpha_{ij}dz^i\wedge d\bar{z}^j$, de classe ($C^0$) avec $\alpha_{ij}h^i\bar{h}^j>0$, telle que $\int_{W^{n-1}\cap K}\alpha\wedge\beta_{n-2}$ soit fini pour tout compact $K\subset D$. On a posé

$$\beta=\tfrac{1}{2}i\sum_k dz^k\wedge d\bar{z}^k;\quad \beta_{n-2}=(1/(n-2)!)\beta^{n-2}.$$

La démonstration part du fait que si V est plurisousharmonique, l'ensemble $V=-\infty$ est de mesure nulle.

b) Un autre résultat concerne la détermination du caractère algébrique d'une variété analytique complexe W^p de C^n à partie d'un critère métrique relatif à l'aire $\sigma_p(r)$ de $W^p\cap B(r)$, où $B(r)$ est la boule $\sum_k|z^k|^2\leqq r^2$ [pour la définition de l'aire cf. également la note du réf., C. R. Acad. Sci. Paris **238**, 2276–2278 (1954); MR **16**, 123] si W^p est de dimension p en tous ses points [au sens de Remmert et Stein, Math. Ann. **126**, 263–306 (1953); MR **15**, 615], si le quotient de $\sigma_p(r)$ à r^{2p} est borné, si enfin il existe dans le plan de l'infini une variété de dimension complexe $n-p-2$ sur laquelle W^p peut être prolongée analytiquement, W^p est algébrique.

P. Lelong (Paris).

Tôki, Yukinari, and Shibata, Kêichi. On the pseudo-analytic functions. Osaka Math. J. **6**, 145–165 (1954).

The authors call a complex-valued function, $f(z)=u+iv$, $z=x+iy$, pseudo-regular if it is continuous on a domain D and if, except for a denumerable relatively closed set, u and v are continuously differentiable and the Jacobian of the mapping $u=u(x, y)$, $v=v(x, y)$ is positive. They develop a concept of the local uniformizer and then prove theorems giving criteria that mappings and limit mappings be interior in sense of Stoilow. They then prove analogues of the maximum modulus theorem, and of the theorems of Picard, Liouville, Schottky, Rouché, Hurwitz and Bloch. The development is surprisingly self-contained and an excellent survey of basic analogues to qualitative function theory.

C. J. Titus (Ann Arbor, Mich.).

Theory of Series

Have

***Zamansky, Marc. La sommation des séries divergentes.** Mémor. Sci. Math., no. 128. Gauthier-Villars, Paris, 1954. 46 pp. 700 francs.

This booklet gives, with some references and without proofs, a summary of results concerning inclusion and consistency among methods for evaluating series. More than half of the space is given to well known theorems involving relations among the methods of Abel, Cesàro, Hölder, Nörlund, Riesz, Euler-Knopp, Borel, Riemann, and Lambert. Emphasis is placed upon the methods of Hurwitz-Silverman-Hausdorff. Tauberian theorems are barely mentioned. Essentially all of the results are given with proofs in the more comprehensive book of Hardy [Divergent series, Oxford, 1949; MR **11**, 25] except for results obtained by the

author and Delange since the book of Hardy was written. These results involve methods determined by functions $g(t)$ which are real and continuous and have bounded variation over $0\leq t\leq 1$ and satisfy the conditions $g(0)=1$ and $g(t)=0$ when $t>1$, a series $\sum u_k$ being evaluable to S if

$$\lim_{x\to\infty}\sum_{k=0}^{\infty} g(k/x)u_k=S.$$

Theorems giving inclusion relations among methods of this form are analogous to theorems giving inclusion relations among Hausdorff methods, the Mellin transform $\int_0^\infty t^{s-1}g(t)\,dt$ playing a role analogous to the moment function of the Hausdorff theory. There is no indication that proofs of the theorems involving the Mellin transforms have been published. The theorems are given without proofs by Zamansky [C. R. Acad. Sci. Paris **233**, 908–910, 999–1001 (1951); **235**, 1094–1096 (1952); **236**, 2291–2293 (1953); MR **13**, 455 (where attention is called to an incorrect theorem); **14**, 865, 1079] and by Delange and Zamansky [ibid. **234**, 1025–1027 (1952); MR **13**, 737].

R. P. Agnew (Ithaca, N. Y.).

Agranovič, M. S. On the consistency of several methods of summation. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 169–194 (1954). (Russian)

Real regular triangular matrices A, generating transformations of the form $T_n=\sum_{k=0}^n a_{nk}s_k$, are treated. It is assumed that $a_{nn}\neq 0$ and hence that the transformations have inverses. In case $T_n\to T$ as $n\to\infty$, the sequence s_k is evaluable A to T and we write $A(s_n)=T$. Numerous theorems, remarks, and examples involve the following concepts. A matrix A is translative if $A(s_n)=A(s_{n+1})$ whenever at least one of $A(s_n)$ and $A(s_{n+1})$ exists as a finite number, and is totally translative if $A(s_n)=A(s_{n+1})$ whenever at least one of $A(s_n)$ and $A(s_{n+1})$ exists as a finite number or $+\infty$ or $-\infty$. The matrix A is strongly translative if there exists an integer r such that each pair $s_0, s_1, s_2, \cdots$ and $s_1, s_2, \cdots$ of real sequences has transforms T_n and T_n' such that $T_n=t_nT'_{n+r}+\epsilon_n$, where $t_n\to 1$ and $\epsilon\to 0$ as $n\to\infty$. Two matrices A and B are equivalent if $A(s_n)=B(s_n)$ whenever at least one of $A(s_n)$ and $B(s_n)$ exists as a finite number, and are totally equivalent if $A(s_n)=B(s_n)$ whenever at least one of $A(s_n)$ and $B(s_n)$ exists as a finite number or $+\infty$ or $-\infty$. They are strongly equivalent if there exists an integer r such that each real sequence s_n has A and B transforms T_n and U_n such that $T_n=t_nU_n+\epsilon_n$, where $t_n\to 1$ and $\epsilon_n\to 0$ as $n\to\infty$. Two matrices A and B are consistent (in the usual sense) if they never assign unequal finite values to the same sequence, that is, if $A(s_n)=B(s_n)$ whenever both $A(s_n)$ and $B(s_n)$ exist as finite numbers. They are totally consistent if $A(s_n)=B(s_n)$ whenever both $A(s_n)$ and $B(s_n)$ exist as finite numbers or $\pm\infty$. The main theorems, in which A and B are assumed to be real regular triangular matrices having inverses, are the following. If A and B are both totally translative, then they are consistent. If A and B are totally regular and totally translative, then they are totally consistent.

R. P. Agnew (Ithaca, N. Y.).

***Andersen, A. F. On summability factors of absolutely C-summable series.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 1–4 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Under the restriction that r and ρ be integers Bosanquet [J. London Math. Soc. **20**, 39–48 (1945); MR **7**, 432] proved the following. If $0\leq\rho\leq r$, then necessary and sufficient conditions that a factor sequence ϵ_n be such that $\sum\epsilon_nu_n$ is evaluable $|C, r-\rho|$ whenever $\sum u_n$ is evaluable $|C, r|$ are (1) $\epsilon_n=O(n^{-\rho})$ and (2) $\Delta^r\epsilon_n=O(n^{-r})$. The present note points out respects in which the problem becomes more difficult when ρ and r are not integers, and gives a theorem to the effect that the conditions (1) and (2) remain sufficient whenever ρ and r are real numbers for which $0\leq\rho\leq r$. There are no proofs, and it is said that a detailed exposition of the subject will appear in Acta Mathematica.

R. P. Agnew (Ithaca, N. Y.).

Chow, Hung Ching. Note on convergence and summability factors. J. London Math. Soc. **29**, 459–476 (1954).

L. S. Bosanquet [same J. **20**, 39–48 (1945); MR **7**, 432] has determined the $|C,\beta|-|C,\alpha|$ convergence factors for natural numbers α, β. The first extension to arbitrary $\alpha, \beta\geq 0$ is due to A. F. Andersen [see the preceding review]. The author independently proves further generalizations. He gives necessary and sufficient conditions, C, for the factors ϵ_n transforming every $\{a_n\}$ of a certain set S into a series $\sum\epsilon_na_n$ summable $|C,\alpha|$ in the following cases. (i) S: The (C,β)-transform t_n^β of $t_n=na_n$ satisfies $\sum n^{-1}\lambda_n|t_n^\beta|<\infty$. C: $\epsilon_n=O(n^{\alpha-\beta}\lambda_n)$ and $\Delta^\beta(n^{-1}\epsilon_n)=O(n^{-\beta-1}\lambda_n)$ (where $0\leq\alpha\leq\beta$ and $n^{-1}\lambda_n>0$ is non-increasing). (ii) S: $t_n^\beta=O(n^q)$ $(q\geq 0)$. C: $\sum n^{\beta-\alpha-1+q}|\epsilon_n|<\infty$ and $\sum n^{\beta+q}|\Delta^\beta(n^{-1}\epsilon_n)|<\infty$ if $0\leq\alpha\leq\beta$; $\sum n^{-1+q}|\epsilon_n|<\infty$ and $\sum n^{\beta+q}|\Delta^\beta(n^{-1}\epsilon_n)|<\infty$ if $\alpha>\beta\geq 0$.

K. Zeller (Tübingen).

Borwein, D. On the absolute summability of Stieltjes integrals. J. London Math. Soc. **29**, 476–486 (1954).

The author gives necessary and sufficient conditions for the factors $k(t)$ with the property "$\int_1^\infty k(t)dx(t)$ is summable $|C,\lambda+1|$ whenever $\int_1^\infty dx(t)$ is bounded (C,λ)." (It is supposed that $\lambda\geq 0$ and $x(t)$ has bounded variation in every finite interval.) The conditions are: (i) $k(t)$ is continuous; (ii) $\int_1^\infty t^{-1}|k(t)|\,dt<\infty$; (iii) there is a number $c\geq 1$ and a function $h(u)$ such that $k(t)=\int_t^\infty (u-t)^\lambda\,dh(u)/\Gamma(\lambda+1)$ for $t\geq c$, and $\int_c^\infty u^\lambda|dh(u)|<\infty$. Sufficient conditions have been given by L. S. Bosanquet [same J. **23**, 35–38 (1948); MR **10**, 112]. [For series analogues see M. Fekete, Math. Termész. Ért. **35**, 309–324 (1917), and the preceding review.]

K. Zeller (Tübingen).

Sunouchi, Gen-ichiro. On the absolute summability factors. Kōdai Math. Sem. Rep. **1954**, 59–62 (1954).

It is familiar that if $\varphi(t)$, with F.S. (Fourier series) $\sum a_n\cos nt$, is B.V. (bounded variation), then $na_n=O(1)$, and hence (*) $\sum a_n(\log n)^{-1-\epsilon}$ is absolutely convergent if $\epsilon>0$; the example $\sum_1^\infty(-1)^{n-1}(2n-1)^{-1}\cos(2n-1)t$ shows that ϵ cannot be replaced by zero. M. T. Cheng has proved, in the case $0<\alpha\leq 1$, that (1) if $\varphi_\alpha(t)$ is B.V., then (*) is summable $|C,\alpha|$, where $\varphi_\alpha(t)$ is the mean value of order α, base 0, of $\varphi(t)$ [Duke Math. J. **15**, 29–36 (1948); MR **9**, 580, 735]. Here the author proves (1) in the case $\alpha>1$. He divides the proof into two stages: (1_a) if the hypothesis in (1) holds, then $\sum_1^n|\Delta\sigma_\nu^\alpha|=O(\log n)$, where σ_n^α is the nth Cesàro mean of order α for $\sum a_n$; (1_b) if the conclusion in (1_a) holds, then the conclusion in (1) holds. The author attributes (1_a) to the reviewer [Proc. London Math. Soc. (2) **41**, 517–528 (1936)], but the result does not appear explicitly in the reviewer's paper, though it is implicit in his analysis; however it is as easy to prove that $na_n=O(1)$ (C,α), which is a stronger conclusion. (1_b) may also be deduced from certain results obtained independently by H. C. Chow [for the case $0<\alpha<1$, ibid. (3) **1**, 206–216

(1951), Lemma 4; MR 13, 340; for the general case, Theorem I of the paper reviewed second above or (with the reviewer's alternative conclusion in (1_a)) Theorem 2 or Corollary 1].

L. S. Bosanquet (London).

Mohanty, R., and Misra, B. On absolute logarithmic summability of a sequence related to a Fourier series. Tôhoku Math. J. (2) 6, 5–12 (1954).

The first author has proved that if $\varphi(t)$, with F.S. $\sum a_n \cos nt$, is B.V., then (*) $\sum a_n(\log n)^{-1}$ is summable $|R, \exp n^\delta, 1|$ for $0<\delta<1$ [Proc. London Math. Soc. (2) 52, 295–320 (1951); MR 12, 822]. Here the authors prove that if $0<\alpha<1$ and $\varphi_\alpha(t)$ is B.V., then (*) is summable $|R, \exp\{(\log n)^{1+(1/\alpha)}\}, 1|$, where $\varphi_\alpha(t)$ is defined as in the preceding review. By applying suitable Tauberian theorems they deduce sufficient conditions for (*) to be absolutely convergent. They also remark that $\varphi(t)$ B.V. does not imply $s_n/\log n$ B.V., since the latter is a Tauberian condition for (*) to be absolutely convergent whenever it is $|R, \log n, 1|$. An example which may be used to illustrate this point is given in the preceding review.

L. S. Bosanquet.

Moustafa, M. D. Convolution of Cesàro methods. J. London Math. Soc. 30, 85–100 (1955).

P. Vermes [J. Analyse Math. 2, 160–177 (1952); MR 14, 745] defined the convolution $C=A*B$ of two sequence-to-sequence methods $A=(a_{nk})$, $B=(b_{nk})$ by

$$c_{nk}=a_{nk}b_{n0}+a_{n,k-1}b_{n1}+\cdots+a_{n0}b_{nk}.$$

In the present paper the author continues the study of matrix methods of summability arising in this manner and proves a number of results of which the following are typical. The convolution of the $(C, 1)$ method with itself r-times includes the (C, r) method. This is a generalization of a result due to Vermes. Let ρ, π and r be positive integers. Then in order that $(C, \pi)*(C, \rho)$ should include (C, r) it is necessary that: $r\leqq\pi+1$, $r\leqq\rho+1$ and $r\leqq\pi+\rho$. If $\min(\rho, \pi)$ is an integer then the preceding inequalities are also sufficient.

V. F. Cowling (Lexington, Ky.).

Rajagopal, C. T. On Tauberian theorems for the Riemann-Liouville integral. Acad. Serbe Sci. Publ. Inst. Math. 6, 27–46 (1954). Please see the note on p. 1337

The author restates the essence of a theorem of Minakshisundaram and Rajagopal [Proc. London Math. Soc. (2) 50, 242–255 (1948), Theorem 2; MR 10, 245], one half of which may be expressed in the form: Theorem I_a. If

$$\Gamma(\alpha)\phi_\alpha(x)=\int_0^x (x-u)^{\alpha-1}\varphi(u)du=O\{W(x)\}$$

and

$$\overline{\underset{0<t<\theta(x)}{\text{bound}}}\,\{\phi_{\alpha'}(x)-\phi_{\alpha'}(x-t)\}=O\{V(x)\},$$

where $\theta=(W/V)^{1/(\alpha-\alpha')}$ and $0\leqq\alpha'<\alpha$, then

$$\phi_{\alpha'}(x)=O_R\{V(x)\}.$$

There are similar results with O_L or o in the conclusion, or with a discrete variable λ_n in place of x. The author uses Theorem I as a basic result from which he deduces a number of substantially familiar Tauberian theorems. For example, Theorem I' is Theorem 1.81 of C.-M. [Chandrasekharan and Minakshisundaram, Typical means, Oxford, 1952; MR 14, 1077], Theorem II is the Hardy-Littlewood-Riesz convexity theorem for fractional integrals (Theorem 1.71 of C.-M.), Theorem III is a limitation theorem for Riesz summability (Theorem 1.61 of C.-M.). Theorem V, which purports to be a specialization of Theorem IV, would, if correctly stated, include a theorem of Karamata [Glas Srpske Akad. Nauka 191, 1–37 (1948); MR 11, 98]. Unfortunately the author has omitted the vital condition that $\theta=W^{1/(\alpha-\alpha')}$ (or $V(x)=1$), and the final conclusion should be $\phi_{\alpha'}(x)=o(1)$. A similar criticism applies to Theorem VIII.

L. S. Bosanquet (London).

Pati, T. A Tauberian theorem for absolute summability. Math. Z. 61, 75–78 (1954).

Suppose that the positive numbers λ_n increase steadily to infinity as $n\to\infty$. The series $\sum_1^\infty a_n$ is said to be absolutely summable (R, λ, r), $r>0$ [or summable $|R, \lambda, r|$] if $\omega^{-r}\sum_{\lambda_n\leqq\omega}(\omega-\lambda_n)^r a_n$ is, as function of ω, of bounded variation for $\omega\geqq\lambda_1$. Theorem: Suppose that $\sum a_n$ is summable $|R, \lambda, r|$. If the two sequences $\{a_n\lambda_n/(\lambda_n-\lambda_{n-1})\}$ and $\{\lambda_n/\lambda_{n+1}\}$ are of bounded variation, then $\sum a_n$ is absolutely convergent.

W. W. Rogosinski (Newcastle-upon-Tyne).

Pennington, W. B. On Ingham summability and summability by Lambert series. Proc. Cambridge Philos. Soc. 51, 65–80 (1955).

Let $\chi_k(t)=(k+1)t\sum_{n<1/t}(1-nt)^k$ $(0<t<1)$, $\chi_k(t)=0$ elsewhere; let $\Lambda=\{\lambda_n\}$ be an increasing sequence of positive numbers tending to infinity. The series $\sum_{n=1}^\infty a_n$ is summable (I, Λ, k), if $\lim_{x\to\infty}\sum a_n\chi(\lambda_n/x)$ exists. Let $g(s)=s/(e^s-1)$, $L(s)=\sum a_n g(\lambda_n s)$. The series $\sum a_n$ is summable (L, Λ), if $\lim_{s\to 0}L(s)$ exists. The (I) and (L) methods are analogous to the Riesz and Abel methods of summation. The author elucidates this analogy by proving theorems corresponding to theorems about the Riesz and Abel method. Typical results are: 1. (I, Λ, k) for $0<k<k'$ implies (I, Λ, k'). 2. If $L(s)$ is convergent for $s>0$, and if $k\geqq 0$, then (I, Λ, k) implies (L, Λ). 3. For $k>0$, (I, Λ, k) implies

$$a_n=o\{[(\lambda_{n+1}/(\lambda_{n+1}-\lambda_n))^k+(\lambda_n/(\lambda_n-\lambda_{n-1}))^k]\log\lambda_n\}.$$

If $\lambda_n=n$, then $a_n=o(n^k)$. 4. (I, Λ, k) is regular for $k>0$, but $(I, \lambda_n=n, 0)$ is not regular. 5. (L, Λ) is regular.

W. H. J. Fuchs (Ithaca, N. Y.).

Korevaar, Jacob. The Riemann hypothesis and numerical Tauberian theorems for Lambert series. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indag. Math. 16, 564–571 (1954).

It is shown that the following unproved Tauberian proposition on Lambert series is equivalent to the Riemann hypothesis on the zeros of the zeta function. Let the Lambert series

$$\sum_{n=1}^\infty a_n\frac{nx}{e^{nx}-1}$$

converge when $x>0$ to a function $f(x)$ which is the restriction of a function analytic at $x=0$. Let $f(x)\to s$ as $x\to 0+$. If the numbers a_n are real, let there correspond to each $\delta>0$ a number $A_1(\delta)$ such that

$$a_n\geqq -A_1(\delta)n^{-1+\delta}\quad(n=1, 2, 3, \cdots),$$

and in case of nonreal a_n, let Re a_n and Im a_n satisfy similar conditions. Then to each $\epsilon>0$ corresponds a constant $K_1(\epsilon)$ such that

$$\left|\sum_{k=1}^n s_k-s\right|\leqq K_1(\epsilon)n^{-\frac{1}{2}+\epsilon}\quad(n=1, 2, 3, \cdots).$$

Numerous related theorems and remarks are given.

R. P. Agnew (Ithaca, N. Y.).

Žak, I. E., and Timan, M. F. On summation of double series. Mat. Sb. N.S. 35(77), 21–56 (1954). (Russian)

Cesàro and Abel transforms of double series are studied in considerable detail. Much of the paper treats problems that arise because a double sequence can be convergent in the Pringsheim sense and nevertheless have a finite set of unbounded rows and columns. Several theorems suppose that a series is evaluable by one method and give additional conditions, less restrictive than the hypothesis that the series has bounded partial sums, which imply that the series is evaluable by another method. Several theorems on absolute evaluability of double series are given; these correspond more closely to theorems on simple series because each double series which converges absolutely must have bounded partial sums. Finally, conditions on the coefficients of a trigonometric double series, and conditions on a function $f(x, y)$ generating a Fourier series, are given which imply that the series are absolutely evaluable by Cesàro and Abel methods. An addition should be made to the 17 references given by the authors: C. N. Moore, Summable series and convergence factors, Amer. Math. Soc. Colloq. Publ., vol 22, New York, 1938. Works cited by Moore give some of the results of the present paper. *R. P. Agnew.*

Yurtsever, B. Eine Note über divergente Reihen. Comm. Fac. Sci. Univ. Ankara. Sér. A. 6, 1–4 (1954). (Turkish summary)

This is an extension from C_1 to C_r of a theorem of the author [same Comm. 5, 1–11 (1953); MR 16, 28].

R. P. Agnew (Ithaca, N. Y.).

Levin, V. I. Estimation of certain numerical series. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 191–194 (1954). (Russian)

If c_n is a nondecreasing sequence, $c_0=1$, $\rho(x)$ $(x\geqq 1)$ has a continuous derivative, $g(x)=x\rho'(x)/\rho(x)\geqq 0$ and nondecreasing, $\lambda>0$, then (1) $\sum c_n^{-\lambda}\rho(c_{n+1}/c_n)\geqq \rho(\xi)/(1-\xi^{-\lambda})$ holds, where $\xi>1$ is the unique solution of (2) $g(x)=\lambda/(x^\lambda-1)$. Equality is attained for $c_n=\xi^n$. The inequality (1) is true under the weaker assumption that $x\rho'(x)$ is $\geqq 0$ and nondecreasing and that (2) has a unique solution. Special cases: $\rho(x)=x^\mu$, $x^{-1}e^x$, $e^{\sqrt{x}}$, $\ln x+\mu$. *K. Zeller* (Tübingen).

Šidák, Zbyněk. A method of investigating monotone sequences. Časopis Pěst. Mat. 79, 135–139 (1954). (Czech)

Several results of the nature of the following are given, a sequence $a_1, a_2, a_3, \cdots$ being called convex when

$$a_n<(a_{n-1}+a_{n+1})/2.$$

If a_n is convex and monotone increasing, then na_n is convex.

R. P. Agnew (Ithaca, N. Y.).

Wright, Fred M. A transformation for *S*-fractions. Proc. Amer. Math. Soc. 5, 888–901 (1954).

The author shows that if

$$P(z)=\sum_{n=0}^{\infty} p_n z^{n+1} \quad\text{and}\quad Q(z)=\sum_{n=0}^{\infty} q_n z^{n+1}$$

are the power series expansions of the terminating continued fractions

$$K_{n=1}^{2k+1}\left[\frac{a_n z}{1}\right] \quad\text{and}\quad K_{n=1}^{2k}\left[\frac{b_n z}{1}\right],$$

respectively, where $a_i\neq 0$, $i=1, \cdots, 2k$, and $b_i\neq 0$, $i=1, \cdots, 2k-1$, then, in order that $P(z)=[p_0+Q(z)]z$, it is necessary and sufficient that there should exist, for each positive integer i less than $2k+2$, a number g_i, with $g_i\neq 0$, $i=1, \cdots, 2k$, $g_{2i+1}\neq 1$, $i=1, \cdots, k-1$, such that $b_1=g_1g_2$, $b_{2i}=-g_{2i}g_{2i+1}$, $i=1, \cdots, k$, $b_{2i+1}=(1-g_{2i+1})g_{2i+2}$, $i=1, \cdots, k-1$, and $a_1=g_1$, $a_2=-g_2$, $a_{2i+1}=g_{2i}(1-g_{2i+1})$, $i=1, \cdots, k$, $a_{2i+2}=-g_{2i+1}g_{2i+2}$, $i=1, \cdots, k-1$. The same holds if the continued fractions do not terminate $(k\to\infty)$. Applications are made to the backward extensions of Stieltjes and Hausdorff moment sequences. *H. S. Wall* (Austin, Texas).

Huzino, Seiiti. Summation of some series containing solutions of the *F*-equation. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 181–186 (1954).

The $F_i(z, \alpha)$, $i=1, 2$, satisfy Truesdell's functional equation $\partial F(z, \alpha)/\partial z=F(z, \alpha+1)$. The author sums the series:

$$\sum_{n=0}^{\infty} F_1(x, \alpha-n)F_2(y, \alpha+n)z^n, \quad \sum_{n=0}^{\infty} F_1(x, \alpha+n)F_2(y, \alpha+n)z^n/n!.$$

The second of these series has also been summed by Duff [C. R. Acad. Sci. Paris 229, 1195–1197 (1949); MR 11, 244].

A. Erdélyi (Pasadena, Calif.).

Borwein, D. On the abscissae of summability of a Dirichlet series. J. London Math. Soc. 30, 68–71 (1955).

Let $1<l_0<\cdots<l_n<\cdots$, $l_n\to\infty$, and suppose that $D\equiv\lim\sup\,(\log n/\log l_n)<\infty$. Let σ_k, $\bar\sigma_k$ be respectively the abscissae of summability (R, l, k) and $|R, l, k|$ of the Dirichlet series $\sum a_n l_n^{-s}$. The author proves that $\bar\sigma_k\leqq\sigma_k+D$ for all $k\geqq 0$. Cf. L. S. Bosanquet [same J. 22, 190–195 (1948); MR 9, 581] and M. C. Austin [ibid. 27, 189–198 (1952); MR 13, 738] for previous results in this connection.

E. Hille (New Haven, Conn.).

Fourier Series and Generalizations, Integral Transforms

Smyrl, J. L. Uniqueness theorems for a class of generalized trigonometrical series. Proc. Amer. Math. Soc. 5, 971–978 (1954).

If $h>0$ is given, let $\{\mu_n\}$ $(n\geqq 0)$, be the increasing sequence of non-negative roots of the equation $x+h\tan x=0$. With $f(x)\in L(-\pi, \pi)$ one associates the trigonometrical series

$$f(x)\sim a_0+\sum_1^{\infty}(a_n\cos nx+b_n\sin nx)=a_0+\sum_1^{\infty}A_n(x),$$

where $a_0=K_0\int_{-\pi}^{\pi}f(t)dt$, $a_n+ib_n=K_n\int_{-\pi}^{\pi}f(t)e^{i\mu_n t}dt$ $(n\geqq 1)$, with $K_0=h/2(1+\pi h)$ and $K_n=2\mu_n/(2\pi\mu_n-\sin 2\pi/\mu_n)$, $n\geqq 1$. These formulae result from formal termwise integration. The relation $a_0=-\sum_1^{\infty} a_n\cos\mu_n\pi$ then holds automatically. The series was first used by Cauchy and recently studied by L. Fejes [Acta Univ. Szeged. Sect. Sci. Math. 11, 28–36 (1946); MR 8, 263] and S. Verblunsky [Bull. Sci. Math. (2) 76, 85–96 (1952); MR 14, 42]. It is known as the F. A. series of f (Fouriersche Reihe der Abkühlung). If $f(x)+a_0\sim a_0+\sum_1^{\infty} A_n(x)$, then $\sum_1^{\infty} A_n(x)$ is called the modified F.A. series, or M.F.A. series, of f. The following uniqueness theorems correspond closely to those for ordinary Fourier series; the proofs too are reduced to the Fourier case. (1) If $\sum A_n(x)$ converges for all x in $\langle-\pi, \pi\rangle$, except perhaps for a denumerable set, to an $f(x)\in L$, then it is the M.F.A. of f. (2) If the upper and lower Abel limits of $\sum A_n(x)$ are finite and integrable in $\langle-\pi, \pi\rangle$, then $\sum A_n(x)$ differs from a

M.F.A. series by a sum of the form $k\sum_1^\infty K_n \sin \mu_n\pi \sin \mu_n x$. (3) If $\sum A_n(x)$ is Abel summable to 0, p.p., then $a_n = b_n = 0$ for all n. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Men'šov, D. E. On the limits of indeterminateness of partial sums of universal trigonometric series. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. 7, 3–33 (1954). (Russian)

In a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 79–82 (1945); MR **7**, 435] the author introduced the notion of a universal trigonometric series and he proved the existence of such series. He calls the series

$$(S) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

universal if, given any measurable function $f(x)$, $0 \leqq x \leqq 2\pi$, not necessarily finite almost everywhere, we can find such a sequence of partial sums S_{n_i} of S that

$$(*) \qquad \lim S_{n_i}(x) = f(x)$$

almost everywhere. Generalizing this notion he calls S a universal series of type $A(F_1, F_2)$, where $F_1(x) \leqq F_2(x)$ if, given any function $f(x)$ satisfying $F_1(x) \leqq f(x) \leqq F_2(x)$ almost everywhere, we can find an $\{n_i\}$ such that we have (*) almost everywhere. The new notion coincides with the old if $F_1(x) = -\infty$, $F_2(x) = +\infty$. In another paper of his the author defines the upper and lower limits in measure of a given sequence of functions $f_1(x), f_2(x), \cdots$ defined in an interval (a, b) [see Trudy Mat. Inst. Steklov. **32** (1950); MR **12**, 254; **15**, 866, the definition is rather long and the reader is referred to the review]. These limits are defined almost everywhere only, and at almost every point are contained between the ordinary upper and lower limits of the sequence. Given functions $F_1(x) \leqq F_2(x)$ in $(0, 2\pi)$, the author calls S a series of type $B(F_1, F_2)$ if for any four measurable functions $\varphi_1(x)$, $\chi_1(x)$, $\chi_2(x)$, $\varphi_2(x)$ defined almost everywhere in (0, 2) and satisfying the inequalities

$$F_1 \leqq \varphi_1 \leqq \chi_1 \leqq \psi_2 \leqq \varphi_2 \leqq F_2$$

there is a series S and a sequence $\{n_i\}$ with the following properties: the upper and lower limits in measure of $\{S_{n_i}(x)\}$ are $\psi_2(x)$ and $\psi_1(x)$ respectively, and the ordinary upper and lower limits of $S_{n_i}(x)$ are almost everywhere the functions $\varphi_2(x)$ and $\varphi_1(x)$ respectively. The main result of the paper is that every series S of type $A(F_1, F_2)$ is necessarily also of type $B(F_1, F_2)$. In particular, every universal series S is necessarily of type $B(-\infty, +\infty)$.

A. Zygmund (Chicago, Ill.).

Salem, R., and Zygmund, A. Some properties of trigonometric series whose terms have random signs. Acta Math. **91**, 245–301 (1954).

The authors consider trigonometrical series of the type

$$(1) \qquad \sum_1^{\infty} \phi_n(t)(a_n \cos nx + b_n \sin nx),$$

where $\phi_n(t)$ are Rademacher functions, and establish new properties holding for almost all t. The first group involves the partial sums $S_N = S_N(t, x)$ of the first N terms of (1). If $c_n^2 = a_n^2 + b_n^2$, $2B_N^2 = \sum_1^N c_n^2 \to \infty$, it is shown that

$$\limsup S_N(2B_N^2 \log\log B_N)^{1/2} = 1$$

for almost all x and t provided that $C_N^2 = O\{B_N^2[\omega(B_N^2)]^{-1}\}$ and the functions $\omega(p)$, $p[\omega(p)]^{-1}$ both increase and $\sum[p\omega(p)]^{-1} < \infty$. Under the same conditions, the distribution function of S_N/B_N tends, for almost all t, to the normal distribution with mean value zero and dispersion 1. The two theorems are analogous, respectively, to Kolmogoroff's law of the iterated logarithm and the central-limit theorem for sums of independent random variables, and show that the conclusions hold for almost all t in spite of the fact that the terms of (1) are merely uncorrelated and not independent.

The next group of theorems are inequalities for the maxima $M_n = M_n(t)$ of the moduli of random trigonometrical polynomials $P_n(x, t) = \sum_1^n r_m\phi_m(t) \cos mx$. If $R_n = \sum_1^n r_m^2$, it is shown that

$$\limsup M_n(t)[R_n \log n]^{-1/2} \leqq A \leqq 2$$

for almost all t and, with certain extra assumptions about r_m, that $\liminf M_n(t)[R_n \log n]^{-1/2} \geqq c > 0$. In particular, if $n\sum_1^n t_m^4 = O[R_n^2]$, we may take $A = 1$, $c = 1/\sqrt{24}$. The inequalities hold also if $M_n(t)$ is the maximum of $|P_n|$ over a fixed subinterval of $(0, 2\pi)$, and similar results are proved for Steinhaus polynomials of the type $\sum r_m \exp[imx + 2\pi i\alpha_m]$, where α_m are independent random numbers in [0, 1]. The maxima $M_n(t)$ of the power polynomials $\sum r_m\phi_m(t)x^m$ are found to behave differently in that they have no definite order of magnitude, for almost all t, as $n \to \infty$.

Finally, it is shown that a sufficient condition for the series $\sum_1^\infty r_m\phi_m(t) \cos mx$ to converge uniformly in $0 \leqq x \leqq 2\pi$ to a continuous sum, for almost all t, is that

$$\sum \left[\sum_{n+1}^{\infty} r_m^2 \right] n^{-1}(\log n)^{-1/2} < \infty$$

This is also a necessary condition provided that r_m decreases and $(\log n)^p \sum_{m>n} r_m^2$ increases $(p > 1)$. Alternative necessary (but not sufficient) conditions are given in other cases.

In establishing their conclusions, the authors give conditions on γ_n for the convergence to 0, for almost all t, of sequences $[\gamma^1\phi_1(t) + \cdots + \gamma_n\phi_n(t)]/S_n$, when $S_n = \gamma_1 + \cdots + \gamma_n$ and ϕ_n are any orthonormal functions. In particular, they give a new proof of the theorem of Maruyama and Tsuchikura [Tsuchikura, Proc. Japan Acad. **27**, 141–145 (1951); MR **13**, 739] that $\gamma_n = o[S_n/\log\log S_n]$ is a sufficient and best possible condition when ϕ_n are Rademacher functions.

H. R. Pitt (Nottingham).

Ul'yanov, P. L. Application of A-integration to a class of trigonometric series. Mat. Sb. N.S. **35**(77), 469–490 (1954). (Russian)

The principal results were announced in Dokl. Akad. Nauk SSSR (N.S.) **90**, 33–36 (1953); MR **15**, 27.

R. P. Boas, Jr. (Evanston, Ill.).

Kahane, Jean-Pierre. Sur les fonctions sommes de séries trigonométriques absolument convergentes. C. R. Acad. Sci. Paris **240**, 36–37 (1955).

Let A denote the class of functions equal, in every neighborhood, to sums of absolutely convergent Fourier series. The author announces the construction of a function $f(x)$ which is in A while $|f(x)|$ is not in A. He has also sharpened a result of Leĭbenzon [Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 157–162 (1954); MR **16**, 241] on functions $\omega(x)$ such that $f(\omega(x))$ is in A if $f(x)$ is. Finally, he gives some classes B and C of functions such that $g(h)$ is in A if g is in B and h is in C. *R. P. Boas, Jr.* (Evanston, Ill.).

Dugué, Daniel. Sur l'approximation d'une fonction caractéristique par sa série de Fourier. C. R. Acad. Sci. Paris **240**, 151–152 (1955).

The author proves the elementary theorem: A characteristic function is represented by its Fourier series on the interval $(-\pi, \pi)$, the convergence being uniform.

H. P. McKean, Jr. (Princeton, N. J.).

Kuzmak, G. E. On a system of functions. Mat. Sb. N.S. 35(77), 461–468 (1954). (Russian)

By using results of N. Bari on sets of functions "close" to normed, orthogonal, complete functions, the author proves the following theorem. Let $0<a<f(x)<b$, $\psi_0(x)=c_0f(x)$,

$$\psi_{2m-1}(x)=c_{2m-1}(1+(f(x)/m))\sin mx,$$
$$\psi_{2m}(x)=c_{2m}(1+(f(x)/m))\cos mx,$$

where the c's are chosen so that the ψ's are normed in $L^2(0, 2\pi)$. Then the ψ's form a basis of $L^2(0, 2\pi)$. The result is applied to obtain a solution (in a suitably general sense) of the integro-differential equation

$$f(x)y(x)+\frac{1}{2\pi}\int_0^{2\pi}\frac{dy}{du}\cot\frac{u-x}{2}du=F(x)$$

as a series $\sum A_n\psi_n(x)$. *W. H. J. Fuchs* (Ithaca, N. Y.).

Džrbašyan, M. M. On the integral representation of functions continuous on several rays (generalization of the Fourier integral). Izv. Akad. Nauk SSSR. Ser. Mat. 18, 427–448 (1954). (Russian)

The author has discussed an L^2-theory of generalized Fourier transforms based on Mittag-Leffler functions in a note [Dokl. Akad. Nauk SSSR (N.S.) 95, 1133–1136 (1954); MR 15, 947] which was written after the paper under review. Here he studies the asymptotic behavior of the Mittag-Leffler functions and discusses the ordinary convergence of some generalized Fourier transforms. In particular, he discusses the integral representation of a function which is continuous on one or more rays issuing from the origin.

R. P. Boas, Jr. (Evanston, Ill.).

Have

***Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. Tables of integral transforms. Vol. II. Based, in part, on notes left by Harry Bateman.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. xvi+451 pp. $8.00.

This is the second of the two volumes of tables of integral transforms. It contains about 2500 formulae. Some of them have been taken from standard literature, some from sources which were not readily accessible; and a great many of them are entirely new. This is not surprising; as even basic facts like the generalisation of the Laplace transformation by the K-transformation and the properties of the latter (see Chapter X) have only been found during the past two decades.

The first part, Chapters VIII–XV, continuing Volume I [MR 15, 868], contains tables of further integral transforms: Hankel and other transforms for which the kernels are Bessel functions in the widest sense, viz. Y-, K-, H- and Kontorovich-Lebedev transforms; fractional integrals; Stieltjes transforms; and Hilbert transforms. No extensive tables appear to exist for any of these transformations, and comparatively few transform pairs have been known for some of them. Here Hankel transforms are given of all kinds of elementary functions, of orthogonal polynomials, of Legendre functions, of various Bessel and of related functions, of hypergeometric functions and generalised hypergeometric series, and of miscellaneous functions (the Fourier cosine and sine transforms of $x^{-\mu}f(x)$, $\mu>0$, which the reviewer had missed in volume I are stated in a generalised form in 8.1). Similarly, for the other transformations, transforms are listed both of elementary and of higher transcendental functions. The reviewer remarks, however: systems of characteristic functions of the integral equation associated with a transformation, e.g. of $\mathfrak{H}_\nu f=\lambda f$ $(\lambda=\pm1)$, have played a considerable part in the literature; but they are not mentioned here. In particular, results on Hilbert transforms (Ch. XV) could be simplified and the reader could be helped in finding further Hilbert transforms (e.g., of any rational function $f(x)$ such that $f(z)$ is regular on $y=0$ and $f(z)\to0$ as $|z|\to\infty$; $z=x+iy$) by referring to the known fact due to Hille and Tamarkin: if $F(z)$ is analytic for $y\geqq0$ or $y\leqq0$ and satisfies some condition of boundedness, then $\mathfrak{H}F=iF$ or $-iF$, respectively, where $\mathfrak{H}F$ is the Hilbert transform of $F(x)$.

The second part of this volume contains various integrals involving higher transcendental functions. Some of them cannot be written as transforms; others were not included in the transform tables and are given here, e.g. formulae in 19.2. The integrals concern orthogonal polynomials, viz. Tchebichef, Legendre, Gegenbauer, Jacobi, Hermite and Laguerre polynomials; the complete and incomplete Γ function and related functions; Legendre functions on finite and infinite intervals; various kinds of Bessel functions, and finally of hypergeometric functions. A wealth of formulae! In the appendix, notations and definitions of higher transcendental functions are given; it is followed by an index of notations. Altogether an enormous amount of work has been done; it will be of great use to mathematicians, since the book presents the vast subject in a clearly arranged form. *H. Kober* (Birmingham).

Doetsch, Gustav. Problems solved and unsolved in the theory of the Laplace transform. Rev. Acad. Ci. Madrid 46, 125–136 (1952). (Spanish)

This is a review article describing briefly what is known in certain directions concerning the convergence and representation properties of the unilateral and bilateral Laplace transforms. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

San Juan Llosá, Ricardo. Charakterisierung der durch einfach konvergente Laplace-Integrale darstellbaren Funktionen. Math. Nachr. 12, 113–118 (1954).

The author establishes necessary and sufficient conditions for a function $f(s)$ to be representable as a Laplace transform $f(s)=\int_0^\infty e^{-st}F(t)dt$ for Re $s>x_0$, where $F(t)$ is Lebesgue integrable over every finite interval. These conditions are essentially that

$$(2\pi i)^{-1}\int_{x-i\infty}^{x+i\infty}e^{st}f(s)s^{-1}ds \quad (x>x_0)$$

should behave like a function of the form $\int_0^t F(u)\,du$.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Bhatnagar, K. P. On a general transform. Ganita 4, 99–122 (1953).

In earlier papers [see MR 14, 977; 15, 216, 790] the author studied the integral transformation the kernel of which is a resultant of two Hankel kernels. He now extends his investigations to a resultant of three kernels. *A. Erdélyi.*

***Janet, Maurice. Précis de calcul matriciel et de calcul opérationnel.** Presses Universitaires de France, Paris, 1954. viii+222 pp. 1800 francs.

The author explains in the preface that this book consists of two disconnected parts. The first of these (50 pp.) deals with finite matrices and some of their applications, while the second part (160 pp.) introduces the reader to a theory of operational calculus based on Laplace transforms. A summary of the contents follows.

Matrices. Linear dependence, determinants, algebra of matrices, singular matrices, linear systems, latent roots, diagonalization, the Hamilton-Cayley theorem, Duncan's formula, symmetric matrices, the orthogonal group, hermitian matrices, the unitary group, differentiation of matrices with application to a system of linear ordinary differential equations.

Operational calculus. I. Generalities. The operator of integration, Laplace integrals, the convolution formula, properties of the convolution, uniform convergence, Fourier integrals, functions of a complex variable. II. The Laplace transformation. Basic properties, convergence and analyticity of Laplace integrals, Lerch's theorem, the complex inversion formula, two-sided Laplace integrals, entire functions of exponential type and their Laplace transforms, sufficient conditions for a function to be a Laplace transform. III. Applications to partial differential equations. Introduction, the one-dimensional diffusion equation, the one-dimensional wave equation, the telegraphist's equation. IV. Operational calculus. Duhamel's formula, the operational image, nomenclature. V. Functionals and distributions. Infinitely differentiable functions with compact support, distributions, derivatives of distributions, convolution of distributions, applications to potential theory. VI. Short table of Laplace transforms. Supplement and examples. List of references.

The author has succeeded in condensing a considerable amount of information in a slim volume. *A. Erdélyi.*

Chakravarty, N. K. Note added to the paper, "On certain theorems in operational calculus with two variables." Ganita 4, 129–130 (1953).

See Ganita 4, 1–11 (1953); MR 15, 120.

Rathie, C. B. A theorem in operational calculus. Ganita 4, 135–137 (1953).

If $\phi(p)$ is the operational image of $f(x)$, and $p^{2-\nu}e^{ap^2}f(p)$ is the operational image of $g(x)$, the author expresses $\phi(p)$ as an integral involving $g(x)$. *A. Erdélyi.*

Gupta, H. C. An operational relation. Math. Z. 61, 70–74 (1954).

The author evaluates the inverse operational image of

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(-s)\Gamma(-\nu s-\rho)\prod_{q=1}^{m}\Gamma(a_q+b_q s)}{\prod_{r=1}^{n}\Gamma(c_r+d_r s)}p^{\nu s+\rho+1}ds$$

under suitable conditions on the parameters.

A. Erdélyi (Pasadena, Calif.).

Šil'krut, D. I. On a generalization of a transformation of Èfros. Prikl. Mat. Meh. 18, 627–630 (1954). (Russian)

Let $F(p)$ be the operational image of $\phi(t)$. Then $u(p)F(p)$ is the operational image of

$$\text{(1)}\qquad \Phi(t)=\frac{d^2}{dt^2}\int_0^t\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{u(p)}{p^2}\phi(\tau)e^{p(t-\tau)}dp\right\}d\tau.$$

If $\phi(t, \lambda, \mu, \cdots)$ depends on certain parameters, so does $F(p, \lambda, \mu, \cdots)$, and (1) remains true. The author points out that (1) remains true if these parameters are replaced by suitable analytic functions of p, and indicates some application of this to the solution of partial differential equations.

A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Parodi, Maurice. Sur une propriété des courbes planes dont le rayon de courbure est une fonction rationnelle de l'abscisse ou de l'ordonnée. C. R. Acad. Sci. Paris 239, 1177–1178 (1954).

Let $R=|p(y)/q(y)|$ where $p(y)$ and $q(y)$ are the real polynomials

$$p(y)=y^n+a_1y^{n-1}+\cdots+a_n,\quad q(y)=b_1y^{n-1}+b_2y^{n-2}+\cdots+b_n.$$

Then the curve $y=y(x)$ has the radius of curvature R at the real zeros of the polynomial $f(y)=y^n+c_1y^{n-1}+\cdots+c_n$, where $c_k=a_k-\epsilon b_kR$ and $\epsilon=\pm1$. As the author indicates, all the zeros of $f(y)$ for which $|y|\leqq|$ also lie in the circle $|y+c_1|\leqq\rho=|c_2|+\cdots+|c_n|$; whereas, if $|c_1|>1+\rho$, there is only one real zero $|y|>1$ and, when n is even, an additional real zero $|y|<1$. *M. Marden.*

Obreschkoff, Nikola. Über die Wurzeln algebraischer Gleichungen mit reellen Koeffizienten. Arch. Math. 5, 506–509 (1954).

Let $f(x)$ be a real polynomial of degree n, let α and β be consecutive real roots of $f(x)=0$, $\alpha<\beta$, with β a q-fold root, let r be the number of roots which are greater than β, and let K be the circle of which $[\alpha, \zeta]$ is a diameter, where $\zeta=\beta-q(\beta-\alpha)/(n-r)$. The author proves the following refinement of Rolle's theorem: If $f(x)=0$ has no roots in the circle K, then the equation $f'(x)=0$ has at least one real root η for which $\alpha<\eta\leqq\zeta$. The case $\eta=\zeta$ is possible only if α and β are the only real roots of $f(x)=0$ and the imaginary roots all lie on the boundary of K. Application is made to the method of Laguerre for the numerical solution of algebraic equations with real and distinct roots.

E. F. Beckenbach (Los Angeles, Calif.).

Villari, Gaetano. Formule asintotiche per gli zeri dei polinomi d'Hermite. Giorn. Ist. Ital. Attuari 15, 93–103 (1952).

Let $x_{n,r}$, $n=0, 1, 2, \cdots$, $r=\pm1, \pm2, \cdots$ be the rth zero of $H_n(x)$. The author proves

$$x_{2n,r}=\frac{(2r-1)\pi}{2(4n+1)^{1/2}}+\frac{A(r)}{(4n+1)^2},$$

$$x_{2n+1,r}=\frac{r\pi}{(4n+3)^{1/2}}+\frac{B(r)}{(4n+3)^2},$$

where

$$0<A(r)<\frac{4}{5}\left(r-\frac{1}{3}\right)^2\pi^2\quad\text{if}\quad\left(r-\frac{1}{3}\right)\pi<(4n+1)^{1/2},$$

$$0<B(r)<\frac{7}{10}\left(r+\frac{1}{12}\right)^2\pi^2\quad\text{if}\quad\left(r+\frac{1}{12}\right)\pi<(4n+3)^{1/2}.$$

A. Erdélyi (Pasadena, Calif.).

Toscano, Emilia. Su alcuni polinomi che al limite si riducono a quelli di Hermite e di P. Humbert. Matematiche, Catania 8, no. 2, 59–72 (1953).

Various known properties of the Hermite, Laguerre, ultraspherical and the so-called Humbert polynomials are established by means of representations of these polynomials as determinants. Several limiting relations are also obtained this way. *G. Szegö* (Stanford, Calif.).

Siraždinov, S. H. On the theory of multidimensional Hermite polynomials. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 70–95 (1949). (Russian)

The author discusses the expansion of an "arbitrary" function in an infinite series of n-dimensional Hermite polynomials [Erdélyi et al., Higher transcendental functions, vol. II, McGraw-Hill, New York, 1953, sections 12.8 to 12.10; MR **15**, 419]. His main result is that the expansion converges provided that the "arbitrary" function is n times continuously differentiable and all partial derivatives up to and including order n are quadratically integrable (with the weight function of the Hermite polynomials as measure).

A. Erdélyi (Pasadena, Calif.).

Palamà, Giuseppe. Polinomi più generali di altri classici e dei loro associati, e relazioni tra essi. Funzioni di seconda specie. Riv. Mat. Univ. Parma **4**, 363–386 (1953).

Let $P_n(x)$, $n=0, 1, 2, \cdots$ be a system of polynomials of degree n satisfying the recurrence formula

$$(1) \qquad P_{n+1}(x)=(a_n+b_nx)P_n(x)-c_nP_{n-1}(x),$$

and let $p_n(x)$ be "functions of the second kind", also satisfying (1). Polynomials $R_{n,p}(x)$ are defined by repeating (1) and writing

$$P_{n+p}(x)=R_{n,p}(x)P_n(x)-c_nR_{n+1,p-1}(x)P_{n-1}(x).$$

(The $R_{n,p}$ are analogous to Lommel's polynomials in the theory of Bessel functions.) The $R_{n,p}(x)$ "generalize" both $P_n(x)$ and $p_n(x)$ insofar as

$$P_n(x)=R_{0,n}(x), \quad p_n(x)=p_0(x)P_n(x)-c_0p_{-1}(x)R_{1,n-1}(x).$$

The $R_{1,n-1}(x)$ are called the associate polynomials.

The author develops the formal theory of the polynomials $R_{n,p}(x)$ and applies his results to the classical orthogonal polynomials and the corresponding functions of the second kind. He shows in particular that relations between two systems of polynomials (for instance, the well-known relation between Laguerre and Hermite polynomials) are mirrored in corresponding relations between the R-polynomials.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Polinomi associati a polinomi classici. Riv. Mat. Univ. Parma **4**, 387–402 (1953).

The associated polynomials referred to in the title are constant multiples of $R_{1,n-1}(x)$ of the preceding review. The author studies these in the cases that the $P_n(x)$ are ultraspherical, Laguerre, or Hermite polynomials.

A. Erdélyi (Pasadena, Calif.).

Olsen, Haakon. On a certain identity in Laguerre polynomials and the related Hankel transform. Arch. Math. Naturvid. **52**, 1–8 (1954).

The author obtains the expansion of a product of two Laguerre polynomials in a series of Laguerre polynomials, and evaluates a Hankel transform closely related to this expansion. [For the integral see Erdélyi et al., Tables of integral transforms, vol. II, McGraw-Hill, New York, 1954, p. 43, equation (8); MR **16**, 468.] *A. Erdélyi.*

Hylleraas, Egil. Expansion of products of Laguerre polynomials. Arch. Math. Naturvid. **52**, 69–72 (1954).

Shorter proof of the expansion mentioned in the preceding review. *A. Erdélyi* (Pasadena, Calif.).

Vermes, P., and Mikhail, M. N. Generated basic sets of polynomials. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indag. Math. **16**, 556–559 (1954).

The authors introduce a simple transformation which in many cases preserves the expansion properties of a basic set of polynomials. In particular, many previous results for simple sets with leading coefficient unity are valid without any restriction on the leading coefficient.

R. P. Boas, Jr. (Evanston, Ill.).

Vidav, Ivan. Sur le problème d'approximation de S. Bernstein et ses généralisations. Acta Math. **91**, 303–316 (1954).

Proofs of previously announced results [C. R. Acad. Sci. Paris **238**, 1959–1961, 2138–2140 (1954); MR **15**, 870, 955].

W. H. J. Fuchs (Ithaca, N. Y.).

Džrbašyan, M. M. Metric criteria of completeness of polynomials for weighted approximation on infinite curves. Dokl. Akad. Nauk SSSR (N.S.) **98**, 713–716 (1954). (Russian)

The author states several theorems dealing with weighted polynomial approximation, in either the uniform or L^2 sense, on the union of a finite number of curves extending to ∞.

R. P. Boas, Jr. (Evanston, Ill.).

Grebenyuk, D. G. On the construction of certain uniform approximations. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 20–29 (1949). (Russian)

The author applies the general theory of approximation in order to work out explicitly the solutions of some problems in one and two dimensions on polynomials of small degree which deviate least from zero under certain restrictions on their coefficients. *R. P. Boas, Jr.* (Evanston, Ill.).

Special Functions

Have

✱**Lense, Josef. Kugelfunktionen. 2te Aufl.** Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1954. xiii+294 pp. DM 26.00.

Reprint of the first edition [1950; MR **12**, 701] with correction of known errors.

Have

✱**Goudet, Georges. Les fonctions de Bessel et leurs applications en physique. 2ème éd.** Masson et Cie, Paris, 1954. 90 pp. 600 francs.

This pamphlet gives a brief introduction to Bessel functions for the use of physicists and engineers. The presentation is as elementary as possible, some results are deduced, others merely described.

Contents: Ch. I, Introduction of Bessel functions through the problem of a vibrating circular membrane. Ch. II. Power series for Bessel functions of the first, second, and third kinds. Graphs. Circular membrane. Ch. III. Recurrence and differentiation formulas, simple integrals, asymptotic expansions, modified Bessel functions. Ch. IV. Applications to electromagnetic waves in a cylindrical wave guide, skin effect, heat conduction in a cylinder, diffraction. Numerical tables. *A. Erdélyi* (Pasadena, Calif.).

Rabinovič, Yu. L. On the functions of Čebyšev-Hermite $H(z, \lambda)$. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 98–117 (1954). (Russian)

The author develops at considerable length the properties of parabolic cylinder functions, without showing any signs

of knowing the existing literature. The first, and longest, part contains differential equations, integral representations, convergent and asymptotic expansions, etc., all known. The second part contains a discussion of the zeros, and asymptotic formulas for the case that the order of the parabolic cylinder function is large. *A. Erdélyi.*

Toscano, Emilia. Relazioni integrali sulla funzione ipergeometrica di Kummer. Matematiche, Catania 8, no. 2, 51–58 (1953).

The author proves the known relation

$$\int_0^1 t^{\gamma-1}(1-t)^{\delta-\gamma-1}{}_1F_1(\alpha;\beta;zt)dt = \frac{\Gamma(\gamma)\Gamma(\delta-\gamma)}{\Gamma(\delta)}{}_2F_2(\alpha,\gamma;\beta,\delta;z),$$

and from this expansion deduces a number of integrals involving Bessel functions, Struve functions, Hermite and Laguerre functions. *A. Erdélyi* (Pasadena, Calif.).

Cherry, T. M. On Kepler's equation. Proc. Cambridge Philos. Soc. 51, 81–91 (1955).

Formulas for the unique real root ξ^* of Kepler's equation, $\theta=\xi-x\sin\xi$, when θ, x are real with $0<x<1$, are well-known [A. Wintner, The analytical foundations of celestial mechanics, Princeton, 1941, pp. 212–222; MR 3, 215]. In this paper analogous expressions for roots with non-vanishing imaginary part are obtained. Integral formulas are given for real roots when $x>1$. The formulas for ξ^* may be considered as Fourier series in θ. The conjugates of these series are summed. *N. D. Kazarinoff* (Lafayette, Ind.).

Harmonic Functions, Potential Theory

Ishikawa, Osamu. On the class S_λ. Proc. Japan Acad. 30, 424–427 (1954).

It is known that the logarithm of a nonnegative function $f(x,y)$ is subharmonic if and only if either of the functions

$$(*)\qquad e^{\alpha x+\beta y}f(x,y)$$

$$(**)\qquad [(x-\alpha)^2+(y-\beta)^2]f(x,y)$$

is subharmonic for all real constants α, β. The author now proves a result which is equivalent to the statement that $\log f(x,y)$ is subharmonic provided there is some positive power μ, $0<\mu<2$, of (*) or (**) which is subharmonic. For $0<\mu\leqq 1$, the result follows immediately from the known result; it actually is valid for all positive μ.

E. F. Beckenbach (Los Angeles, Calif.).

Steiner, Antonio. Ueber einen Fall harmonischer Fortsetzung im Grossen. Rend. Circ. Mat. Palermo (2) 3, 198–213 (1954).

Let $u(x,y)$ be bounded and harmonic in the half-plane $y>0$, continuous in $y\geqq 0$; set $g(x)=u(x,0)$, and assume that $g'(0)$ exists as a finite number. Let $D>0$ be given. The following two conditions are shown to be equivalent. (A) $u(x,y)$ can be extended to a function harmonic in $y>-D$, bounded in $y\geqq\delta$ for every $\delta<D$. (B) If $0<\delta<D$ and $\gamma_1(\alpha)$, $\gamma_2(\alpha)$ are the Fourier cosine and sine transforms, respectively, of $x^{-1}[g(x)-g(0)]$, then $e^{\delta\alpha}\gamma_1(\alpha)$ and $e^{\delta\alpha}\gamma_2(\alpha)$ belong to $L^2(0,\infty)$, and

$$\int_0^\infty e^{\delta\alpha}[\gamma_1(\alpha)\cos\alpha x+\gamma_2(\alpha)\sin\alpha x]d\alpha$$

is of order $O(|x|^{-1})$. *W. Rudin* (Rochester, N.Y.).

Ozawa, Mitsuru. On a maximality of a class of positive harmonic functions. Kōdai Math. Sem. Rep. 1954, 65–70 (1954).

Continuation of anterior studies of the author on extended C-ends [same Rep. 1954, 33–37, 55–58, 70; MR 16, 245]. Let Ω denote an extended C-end with non-compact relative boundary Γ. Let Q_0 denote the family of positive harmonic functions w on Ω which vanish continuously on Γ and satisfy $\int_\Gamma \partial w/\partial n\, ds<+\infty$. Let G_0 denote the family of positive harmonic functions on Ω which are representable as linear combinations with positive coefficients of limit functions of the Green's function of Ω (pole tending to the ideal boundary). It is shown that $G_0=Q_0$. *M. Heins.*

***Myrberg, Lauri. Über die Existenz von positiven harmonischen Funktionen auf offenen Riemannschen Flächen.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 214–216 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Remarks on the existence of non-constant positive harmonic functions on Riemann surfaces with positive ideal boundary. It is observed that if a Riemann surface with positive ideal boundary has at least two boundary elements then there exist non-constant positive harmonic functions on the surface. *M. Heins* (Providence, R. I.).

Miles, E. P. Three dimensional harmonic functions generated by analytic functions of a hypervariable. Amer. Math. Monthly 61, 694–697 (1954).

The author considers an algebra determined by the basis 1, ω, ω^2, where $\omega^3=1$, and functions $(\omega-1)f(t)$, where $f(t)$ is an analytic function of $t=x+\omega y+\omega^2 z$. It is shown that any such function is harmonic, although $f(t)$ itself is not necessarily so. This result is consistent with the von Beckh-Widmanstetter theorem that there are no three-unit harmonic algebras, because $\omega-1$ is a nilfactor and annihilates the non-harmonic part of $f(t)$. Thus the class of functions generated is a linear transformation of the two-dimensional harmonic functions, which the author shows by exhibiting the transformation explicitly. *P. W. Ketchum.*

***Trjitzinsky, W. J. Les problèmes de totalisation se rattachant aux laplaciens non sommables.** Mémor. Sci. Math., no. 125. Gauthier-Villars, Paris, 1954. 93 pp. 1400 francs.

Given a real-valued function $f(x,y)$ defined in a plane domain D (f not necessarily summable), the problem considered by the author is to find a continuous function F such that $\nabla^0 F(x,y)=f(x,y)$ in D, where ∇^0 is some generalized Laplacian. The analogous problem in one variable (to find a second primitive of a given function) has been treated by Denjoy and was motivated by uniqueness problems in the theory of trigonometric series. It is to be expected that the extension to two variables (when sufficiently developed) can be similarly applied to the study of Laplace series, but this is beyond the scope of the book under review.

After establishing some theorems about derivatives of potentials, several types of generalized Laplacians are defined and some of their topological properties, as well as topological properties of the derivatives of continuous interval functions, are studied by category arguments. With each $F\in C'$ there is associated a flux $J(F;I)=(\partial F/\partial n)ds$, where I is a rectangle whose boundary is the path of integration; the equation $\nabla^0 F=-\mathrm{Der}\,J(F;I)$, valid except on a certain exceptional set, relates the original problem to that

of finding an interval function whose derivative is f. Several operators and classes of functions are then introduced and their interrelations are studied. Among the tools used is a decomposition of non-summable functions, due to Brelot [C. R. Acad. Sci. Paris **201**, 1316–1318 (1935); **202**, 100 (1936)], and there are numerous references to the work of Denjoy [Leçons sur le calcul des coëfficients d'une série trigonométrique, Gauthier-Villars, Paris, 1941, 1949; MR **8**, 260; **11**, 99]. *W. Rudin* (Rochester, N. Y.).

Ertel, Hans. Ein Theorem über die Feldstärke in Potentialfeldern. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1954**, no. 2, 11 pp. (1954).

Veranschaulicht man ein Potentialfeld durch die Äquipotentialflächen $\Phi(x_1, x_2, x_3)=\text{const.}$ und deren orthogonale Trajektorien (Lotlinien, Kraftlinien), so ergeben sich in jedem (regulären) Feldpunkt $P(x_1, x_2, x_3)$ zum Kraftliniengradienten in P zwei weitere geometrisch ausgezeichnete Richtungen, die Hauptrichtungen der beiden Normalschnitte der Äquipotentialfläche in P. Sind $1/R_1$ und $1/R_2$ die beiden Hauptkrümmungen in P und $1/R_3$ die Krümmung der Kraftlinie durch P in P, so kann man sich die Übungsaufgabe stellen, die relative Abweichung $\Delta F/F$ des absoluten Betrags F des Gradienten $-\text{grad}\,\Phi$ von ihrem Mittelwert $F^{(m)}$ in einer infinitesimalen Umgebung durch diese Krümmungen auszudrücken. Verfasser erhält

$$\frac{\Delta F}{F}=\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}\quad\left(\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right).$$

M. Pinl (Köln).

Have

★**Birkhoff, Garrett. Induced potentials.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 88–96. Academic Press Inc., New York, 1954. $9.00.

The problem of virtual mass, and of a conductor in an uniform field, are extreme cases of a problem having applications in (a) Poisson's theory of induced magnetism, (b) Faraday's theory of induced electrostatic fields, (c) electrical conduction, (d) thermal conduction, (e) percolation, (f) the initial acceleration of a liquid globule in an accelerated medium. The author has coined the phrase "induced potential" to suggest the fact that a single mathematical problem is involved in all these cases. The basic mathematical problem is that of finding the solution $u(\mathbf{x})$ of an elliptic partial differential equation (*) $\nabla\cdot(\sigma\nabla u)=0$, where $\sigma(\mathbf{x})$ is a given function, having a given asymptotic behavior at infinity, and other specified singularities. If σ is constant on confocal ellipsoidal surfaces, a solution suggested by Poisson's solution can be found. In the case of fluids occupying bounded regions the solution of (*) minimizes the integral (**) $\int\sigma(\nabla u\cdot\nabla u)dR$ for given u or $\sigma\partial u/\partial n$ on the boundary. The integral (**) diverges. But for the "polarization field" $\mathbf{a}=\sigma\nabla u-\mathbf{q}$ a corresponding principle can be stated in case $\sigma=1$ outside a sufficiently large sphere. The polarization field maximizes the integral $(\rho\mathbf{a}, \mathbf{a})=\int\rho\mathbf{a}\cdot\mathbf{a}dR$ relative to all divergence-free fields $\mathbf{v}-\mathbf{q}=\mathbf{w}$ which satisfy the induced "polarization energy conservation law":

$$\int\rho\mathbf{w}\cdot\mathbf{w}dR=\int(\rho-\rho_0)\mathbf{q}\cdot\mathbf{w}dR.$$

Finally, the author discusses the discontinuous case of the problem of induced potentials by transformation into a Fredholm integral equation of the second kind. Some integral equations for the discontinuous case have been derived by R. Ingraham, but not published. *M. Pinl.*

Differential Equations

★**Petrowski, I. G. Vorlesungen über die Theorie der gewöhnlichen Differentialgleichungen.** B. G. Teubner, Verlagsgesellschaft, Leipzig, 1954. 198 pp. DM 7.80.

Translation of Petrovskiĭ's Lekcii po teorii obyknovennyh differencial'nyh uravneniĭ [4th ed., Gostehizdat, Moscow, 1952; see MR **12**, 334 for the 3rd ed.].

Agostinelli, Cataldo. Sistemi di equazioni differenziali normali del I ordine che ammettono speciali relazioni invarianti e che interessano il movimento di sistemi anolonomi. Boll. Un. Mat. Ital. (3) **9**, 136–141 (1954).

Systems of the form $\dot{x}_i=F_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, are considered in which the F's satisfy identities of the form

$$\sum_{i=1}^{n-h}\frac{\partial F_{n-h+j}}{\partial x_i}F_i\equiv 0,\quad h<\tfrac{1}{2}n,\quad j=1, 2, \cdots, h.$$

This obviously implies that, if $F_{n-h+1}, \cdots, F_n$ vanish initially for a certain solution, they must vanish identically in t; the $x_{n-h+i}(t)$ reduce to constants for $i=1, \cdots, h$, and the original system reduces to one of order $n-2h$ for the determination of the other x's, at least if a certain jacobian is not zero. The author is also concerned with functions $\varphi_r(x_1, \cdots, x_n)$, $r=1, \cdots, k\leqq h$, which are invariants of such solutions, and his main result is formulated as a necessary as well as a sufficient condition. *D. C. Lewis, Jr.*

Hayashi, Kyûzô. On transformations of differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 313–325 (1954).

Consider the real differential system

$$(1)\qquad dy_i/dx=f_i(x, y_1, \cdots, y_n)\quad (i=1, 2, \cdots, n),$$

where $f_i(x, y)$ are continuous in E_{n+1}: $0\leqq x\leqq a$, $|y_i|<\infty$ $(i=1, 2, \cdots, n)$. The author considers conditions under which the solution curves $y(x)$ approach limit values when $x\to a$. Typical is the following theorem. A necessary and sufficient condition for every solution of (1) to have an endpoint, whose x-coordinate is equal to a, is that, given any positive number α there exists a positive number $\beta(\alpha)$ such that, for any solution $y=y(x)$ of (1) through a point (x_0, y_0) arbitrary in E_{n+1}, provided that $|y(x_0)|\leqq\alpha$, we have $|y(x)|<\beta(\alpha)$ so long as $y=y(x)$ lies in E_{n+1} for $x_0\leqq x\leqq a$.

The author notes also that by compactifying the y-space into the sphere S^n and by considering the image of (1) on S^n, the existence of limit values of the solutions is equivalent to the uniqueness of the critical-point solution at the pole (the point added in compactification) of S^n.

L. Markus (New Haven, Conn.).

Litzman, Otto. The Peano function and orbital stability of a differential equation of the first order. Publ. Fac. Sci. Univ. Masaryk **1953**, 65–90 (1953). (Czech. Russian and French summaries)

The author considers $y'=f(x, y, s)$, where f is continuous, with special reference to cases of non-uniqueness. From the totality of solutions through $x=\xi$, $y=\eta$, for some set M of s-values, there results the Peano function $\Pi(x; \xi, \eta; M)$. A series of results of standard type [cf. E. Kamke, Differentialgleichungen reeller Funktionen, Akademische Verlagsgesellschaft, Leipzig, 1930, Chapter II], leads up to a result on the continuity of $\Pi(x; \xi, \eta; g(\xi, \eta))$, where $g(\xi, \eta)$ is a variable set of s-values. Sets of conditions are then found for uniqueness, and for orbital stability; these conditions

restrict, inter alia, the sign of the variation of f for small displacements. *F. V. Atkinson* (Ibadan).

Myškis, A. D., and Grinfel'd, U. K. On the continuous dependence of the solution of Cauchy's problem upon the initial data. Uspehi Mat. Nauk (N.S.) **9**, no. 3(61), 171–174 (1954). (Russian)

Let $y'=f(x,y)$ have a unique solution curve $\varphi(x;x_0,y_0)$ through each point $a\leqq x_0\leqq b$, $-\infty<y_0<\infty$. Then the author shows that $\varphi(x;x_0,y_0)$ is continuous simultaneously in all arguments. The extension to a first-order partial differential equation is indicated. Finally, the author constructs an example of a second-order equation $y''=F(x,y,y')$ (with discontinuous function F) which satisfies existence and uniqueness conditions for the initial value problem but whose solutions do not depend continuously on the initial values. *L. Markus* (New Haven, Conn.).

Èl'sgol'c, L. È. Remark on the estimation of the number of points of rest of dynamical systems with retarded argument. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 221–222 (1954). (Russian)

The author [Mat. Sb. N.S. **19**(61), 237–238 (1946); **26**(68), 215–223 (1950); MR **8**, 525; **11**, 671] dealt with the same question for a system

$$\dot{x}=f(x), \tag{1}$$

where x, f are n-vectors on a compact n-manifold M^n. Take the system with retarded arguments on M^n:

$$\frac{dx_i(t)}{dt}=f_i(x_1(t-\tau_{11}(t)),\cdots,x_1(t-\tau_{1m_1}(t)),\cdots,\\ x_n(t-\tau_{n1}(t)),\cdots,x_n(t-\tau_{nm_n}(t))), \tag{2}$$

where the right-hand sides are continuous and satisfy a Lipschitz condition. The initial set E_{t_0} is by definition the set of points t_0 and points $t-\tau_{js}(t)<t_0$ for $t\geqq t_0$. A point of rest of (2) is a (constant) point x^0 which is a solution for $t\geqq t_0$ and lies in E_{t_0}. Let $\varphi_i(x(t))$ be what f_i becomes when one replaces every $\tau_{js}(t)$ by zero. Then the points of rest of (2) are the same as those of $\dot{x}=\varphi(x)$ and so amenable to the author's previous enumeration. *S. Lefschetz.*

Zubov, V. I. On the theory of A. M. Lyapunov's second method. Dokl. Akad. Nauk SSSR (N.S.) **99**, 341–344 (1954). (Russian)

Suppose that $\dot{x}=f(x;t)$, where x, f are n-vectors, satisfies for all x, t the usual existence and uniqueness conditions and that, moreover, $f(0;t)=0$ for all t. Let the origin be asymptotically stable. The region A of asymptotic stability of the origin consists of the points $(x^0;t^0)$ of the $(x;t)$-space for which there exists an $\alpha>0$ such that if

$$\sum(x_i^0-\bar{x}_i^0)^2+(t^0-\bar{t}^0)^2<\alpha$$

then $\sum x_i^2(t;\bar{x}^0)\to 0$. The region A is an open and connected set whose boundary [Erugin, Prikl. Mat. Meh. **15**, 227–236 (1951); MR **12**, 705] consists of complete trajectories. In the present paper a number of theorems are stated (without proof) giving necessary and sufficient conditions for asymptotic stability and also the equation of the boundary of A. *S. Lefschetz* (Mexico, D. F.).

Gradštein, I. S. On continuous dependence of asymptotic stability upon a parameter. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 163–166 (1954). (Russian)

Systems of the form (1) $\dot{x}=h(x,p)$, where x, h are n-vectors and p is an l-vector parameter, and their singular points as regards their dependence upon p have recently been studied in a number of papers by the author [see Mat. Sb. N.S. **32**(74), 263–286 (1953); Dokl. Akad. Nauk SSSR (N.S.) **65**, 789–792 (1949); **82**, 5–8 (1952); Uspehi Mat. Nauk N.S. **6**, no. 6(46), 156–157 (1951); MR **14**, 1085, 1086; **10**, 708; **13**, 557]. A first observation is that, even with h very smooth, the system may be asymptotically stable for some p^* but not so outside of p^* in a certain neighborhood of p^* in the space of p. This is the case for instance for $p=0$ and the origin for the following system differing but little from the one discussed by Poincaré:

$$\dot{x}=x(p^2-r^2)+y(p^2+r^2),\quad r^2=x^2+y^2.\\ \dot{y}=y(p^2-r^2)-x(p^2+r^2),$$

If for p asymptotic stability is provable by Lyapunov's first method then it does hold for a certain neighborhood of p. A rather precise theorem, too complicated to be given here, is proved in the paper. *S. Lefschetz.*

Krasovskii, N. N. Sufficient conditions for stability of solutions of a system of nonlinear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **98**, 901–904 (1954). (Russian)

This paper is first concerned with a rectification of a result of Zubov [Prikl. Mat. Meh. **17**, 506–508 (1953); MR **15**, 126]. The assertion was made there that in order that the origin be stable for (1) $\dot{x}=X(x)$, where x, X are n-vectors with X of class C^1 everywhere, it was sufficient that the symmetrized Jacobian matrix J have non-positive characteristic numbers $\lambda_i(x)$. Let S_r denote the sphere $\sum x_i^2=r^2$ and let on S_r:

$$m^2(r)=\inf\sum X_i^2,\quad \lambda(r)=\inf\{|\lambda_i(x)|\}.$$

Then, Theorem 1: In order that S_R be contained in the region of attraction G of the origin, it is sufficient that the $\lambda_i(x)$ be negative within and on S_{R_1}, where if $2N=\sup\sum X_i^2$ on S_R then R_1 satisfies $\int_R^{R_1}m(r)\lambda(r)dr>N$. As a consequence, Theorem 2: Sufficient conditions for the asymptotic stability of the origin in the whole space is that the $\lambda_i(x)$ be negative for all x and that $\int_0^\infty m(r)\lambda(r)dr=\infty$. The author proves also Theorem 3: Let A be a symmetric constant matrix with positive characteristic roots. Suppose that the symmetrized product $A\cdot\|\partial X_i/\partial x_j\|$ has negative characteristic roots in some region containing the origin (origin excluded). Then (1) is asymptotically stable at the origin. *S. Lefschetz* (Mexico, D. F.).

Krasovskii, N. N. On the inversion of theorems of A. M. Lyapunov and N. G. Četaev on instability for stationary systems of differential equations. Prikl. Mat. Meh. **18**, 513–532 (1954). (Russian)

Let there be given the system (1) $\dot{x}=X(x)$, where x, X are n-vectors such that $X(0)=0$ and the components of X are of class C^1 in a certain neighborhood of the origin. The Lyapunov theorem on asymptotic stability was inverted by Massera [Ann. of Math. (2) **50**, 705–721 (1949); MR **11**, 721] and Barbašin [Mat. Sb. N.S. **29**(71), 233–280 (1951); MR **13**, 756]. The present paper deals primarily with the inversion of Lyapunov's instability theorems and complement by Četaev. The basic result refers to the construction of a Lyapunov function $v(x)$ of class C^1 near the origin. Explicitly: Theorem 1: A necessary and sufficient condition for the existence of a $v(x)$ whose time derivative $\dot{v}$ (as a consequence of 1) is of fixed sign near 0 is the existence of a neighborhood U of 0 which does not contain any full trajectory of (1) other than 0. The proof is by methods closely related to those of the papers of Massera and Barbašin.

Th. 1 is now applied to the inversion of Lyapunov's first instability theorem, yielding Theorem 2: Let 0 be unstable. Then the condition of Th. 1 is likewise necessary and sufficient for the existence of a $v(x)$ in the neighborhood of 0 which takes positive values arbitrary near 0 and such that $\dot{v}$ is of fixed positive sign near 0.

Application to the "structural stability" of instability is embodied in the Th. 3. Let 0 be unstable for (1) and let the condition of Th. 1 be fulfilled. There exists then a function $\eta(x)$ $(\eta(0)=0)$ positive near 0 and such that if the vector $R(x;t)$ is of magnitude $>\eta(x)$, then 0 is likewise unstable for the system $\dot{x}=X(x)+R(x;t)$. The proof is patterned after that of Barbašin [Th. 5.2, loc. cit.] on the structural stability of asymptotic stability.

A further instability result deals with Lyapunov's second instability theorem: 0 is unstable if one may write near 0: $\dot{v}=\lambda v(x)+w(x)$ where λ is a positive constant, v is of class C^1 near the origin and w is non-negative there. The author shows that this is also a necessary condition and further that the region where $v>0$ coincides with the region of instability. *S. Lefschetz* (Mexico, D.F.).

Ryabov, Yu. A. Generalization of a theorem of A. M. Lyapunov. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. 7, 131–150 (1954). (Russian)

Lyapunov investigated systems of the form

$$(1)\quad \begin{aligned} &\dot{x}=-\mu y+X, \quad \dot{y}=\mu x+Y,\\ &\dot{x}_s=\sum a_{sj}x_j+ax+by+X_s \quad (s=1,2,\cdots,n), \end{aligned}$$

where μ, a, b, a_{sj} are real constants and X, Y, X_s are holomorphic functions of x, y and the x_s at the origin of the form $[x,\cdots,x_n]_2$ (power series beginning with terms of degree ≥ 2). He proved in particular [Problème générale de la stabilité du mouvement, Princeton, 1947; MR **9**, 34] the following theorem: Suppose that (a) (1) has a formal solution

$$(2)\quad \begin{aligned} x&=cx^{(1)}+c^2x^{(2)}+\cdots,\\ y&=cy^{(1)}+c^2y^{(2)}+\cdots,\\ x_s&=cx_s^{(1)}+c^2x_s^{(2)}+\cdots, \end{aligned}$$

where c is an arbitrary constant whose coefficients $x^{(k)},\cdots$, are periodic in t, which for $k>1$ vanish for $t=t_0$; (b) the matrix $\|a_{jk}\|$ has no characteristic roots zero nor a multiple of $i\mu$. Then the series (2) converge absolutely for sufficiently small c and (2) represents then an actual periodic solution of (1). Lyapunov asserts (but does not prove) that the restrictions under (b) are in fact unnecessary. The main object of the present paper is to prove this assertion. The actual system treated is the one to which Lyapunov reduced (1):

$$(3)\quad \frac{dz_0}{d\theta}=Z_0,\quad \frac{dz_s}{d\theta}=\lambda_s z_s+\sigma_{s-1}z_{s-1}+\varphi_s z_0+Z_s,$$

where the σ_s are certain constants, the λ_j are the characteristic roots of $\|a_{jk}/\mu\|$, the φ_s are quadratic forms in $\sin\theta$, $\cos\theta$, and the Z_s are holomorphic functions of the z_s at the origin with power series $[z,\cdots]_2$ which have for coefficients polynomials in $\sin\theta$ and $\cos\theta$. The solutions are periodic solutions of θ with period 2π.

Consider now a system

$$(4)\quad \frac{dx}{d\theta}=X,\quad \frac{dx_s}{d\theta}=\sum p_{sr}x_r+q_s x+X_s,$$

where p_{sr}, q_s are bounded continuous and periodic with period 2π in θ, and the X, X_s behave like the Z, Z_s save that the coefficients of the series are bounded continuous and periodic with period 2π in θ. If λ_s are the Lyapunov numbers of the solutions of the system

$$\frac{dZ_s^0}{d\theta}=\sum p_{sr}z_r^0,$$

then there is a linear transformation with periodic coefficients reducing (4) to

$$(5)\quad \frac{dx}{d\theta}=\bar{X},\quad \frac{dz_s}{d\theta}=\lambda_s z_s+\sigma_{s-1}z_{s-1}+\varphi_s x+Z_s,$$

with evident meaning of the terms. For (4) also via (5) a theorem like the one of Lyapunov is proved.

Suppose now that in (4) $X=0$. Then $x=\epsilon$, a constant, and (4) takes the form

$$(6)\quad \frac{dx_s}{d\theta}=X_s(\theta,x_1,\cdots,x_n,\epsilon),$$

where X_s behaves as before. Then the theorem already established shows that if (6) admits a formal solution

$$(7)\quad x_s=\epsilon^r x_s^{(1)}+\epsilon^{2/r}x_s^{(2)}+\cdots,$$

with periodic coefficients, then for ϵ small (7) is an actual solution. *S. Lefschetz* (Mexico, D. F.).

Sugiyama, Shohei. Note on singularities of differential equations. Kōdai Math. Sem. Rep. **1954**, 81–84 (1954).

Critical points of the system $\dot{x}=p(x,y)$, $\dot{y}=q(x,y)$ are studied. The system is written $\dot{z}=f(z,\bar{z})$ $(z=x+iy)$, $f(z,w)$ being assumed analytic near the critical point, which is taken as zero. If the series for $f(z,\bar{z})$ starts with nth degree terms, let $f_n(z,\bar{z})$ be the terms of nth degree. Then the index of the critical point at zero is $n-2k$, where k is the number of zeros of $f_n(1,z)$ in $|z|<1$. The equation is topologically equivalent to $\dot{z}=z^{n-2k}$ if $n-2k\geq 0$; otherwise to $\dot{z}=\bar{z}^{2k-n}$. Infinity is handled by stereographic projection. The cases $n=1$, 2, 3, and 4 are considered separately. If $n=1$, and $f_1(z,\bar{z})=az+b\bar{z}$, the index at $z=0$ is $+1$ if $|a|>|b|$, and is -1 if $|a|<|b|$. For $n>1$, the nested-oval or multiple saddle-point form of the characteristics near the critical point, together with separatrices is derived.

W. S. Loud (Minneapolis, Minn.).

Yoshizawa, Taro. On the non-linear differential equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 133–141 (1954).

A general method is given for proving that every solution of $\dot{x}=f(t,x,y)$, $\dot{y}=g(t,x,y)$, is ultimately bounded i.e. satisfies $|x(t)|\leq A$, $|y(t)|\leq B$ for $t\geq t_0$ with constants A, B which depend only on f and g, not on the particular solution. The method depends on the construction of a function $\Phi(x,y)>0$, vanishing at infinity, which (roughly) increases along trajectories outside a large rectangle $|x|<A_0$, $|y|<B_0$. As an example, the method is applied to obtain a boundedness theorem due to the reviewer [J. London Math. Soc. **27**, 48–58 (1952); MR **13**, 844] for the equation

$$\ddot{x}+F(\dot{x})+g(x)=p(t).$$

G. E. H. Reuter (Manchester).

Yoshizawa, Taro. On the convergence of solutions of the non-linear differential equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 143–151 (1954).

A general method is given for proving, for any pair of solutions $(x_1(t),y_1(t))$ and $(x_2(t),y_2(t))$ of the system (S): $\dot{x}=f(t,x,y)$, $\dot{y}=g(t,x,y)$, that $x_2(t)-x_1(t)$ and

$y_2(t)-y_1(t)\to 0$ as $t\to+\infty$. It depends on the construction of a function $\Phi(t, x_1, y_1, x_2, y_2)$ which decreases whenever (x_1, y_1) and (x_2, y_2) move simultaneously along trajectories of (S). This method was used by the reviewer [J. London Math. Soc. **26**, 215–221 (1951); MR **13**, 237] for the special system (S) derived from $\ddot{x}+kf(x)\dot{x}+g(x)=kp(t)$ by putting $y=\dot{x}+k\int_0^x f(u)\,du$. *G. E. H. Reuter.*

Yoshizawa, Taro. Note on the existence theorem of a periodic solution of the non-linear differential equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 153–159 (1954).

The author remarks that if each solution of $\dot{x}=f(t, x, y)$, $\dot{y}=g(t, x, y)$ is bounded for $0\leqq t<\infty$ and if f, g have period ω in t, then by a theorem of J. L. Massera [Duke Math. J. **17**, 457–475 (1950); MR **12**, 705] there is at least one solution of period ω. He then modifies the method of a previous paper [see second preceding review] to obtain conditions for the boundedness of each solution (a weaker conclusion than "ultimate boundedness" in which the bounds are independent of the particular solution but hold only for $t\geqq t_0$). The modified method is used to derive the existence of a periodic solution for $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$ under the conditions given by the reviewer [Proc. Cambridge Philos. Soc. **47**, 49–54 (1951); MR **12**; 827] and by S. Mizohata and M. Yamaguti [Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 109–113 (1952); MR **14**, 874]. The reviewer remarks that this last equation in fact has ultimately bounded solutions, and that non-trivial examples of boundedness of each solution without ultimate boundedness do not seem to be known. *G. E. H. Reuter* (Manchester).

Halanay, A. Solutions presque-périodiques de l'équation de Riccati. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **4**, 345–354 (1953). (Romanian. Russian and French summaries)

The author investigates the Riccati equation

$$y'+y^2+p(t)=0$$

on $-\infty<t<\infty$ when $p(t)$ is almost periodic (a.p.). (i) If $u(t)$ and $v(t)$ are two solutions and $\inf[v(t)-u(t)]>0$, then u v are a.p. (ii) If u and v are a.p. solutions then $\inf[v(t)-u(t)]>0$, and the mean value of $u+v$ is zero. (iii) If $p(t)\leqq 0$ there exist two a.p. solutions. (iv) If the Riccati equation possesses two a.p. solutions, then $x''+p(t)x=0$ possesses two solutions of the form

$$\exp\left[\pm\mu t+\int\alpha_{1,2}(t)dt\right],$$

where α_1, α_2 are a.p. functions with mean value zero.
A. Erdélyi (Pasadena, Calif.).

Gorbunov, A. D. Some questions of the qualitative theory of ordinary linear homogeneous differential equations with variable coefficients. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 39–78 (1954). (Russian)

The author studies real homogeneous linear differential systems (1) $dx/dt=L(t)x$, where

$$x=(x_1, \cdots, x_n),\quad -\infty<t<+\infty,$$

and $L=\|l_{ik}\|$ is a $n\times n$ matrix whose coefficients are continuous functions of t. According to A. M. Lyapunov, given a function $V(x, t)$, by its derivative $V'(t)$ with respect to t relatively to (1) is meant the expression in x and t which is obtained by formal total differentiation of V with respect to t when the derivatives $x_i'=dx_i/dt$, $i=1, \cdots, n$, are replaced by the corresponding linear expressions given by (1). The existence of a definite positive function V whose derivative V' above is non-positive is known to be sufficient (A. M. Lyapunov) to assure that the solution zero of (1) is stable (in the sense of Lyapunov). Other conditions of the same type have been given by I. G. Malkin [see Amer. Math. Soc. Transl. no. **20** (1950); MR **12**, 181] who has also investigated their necessity. The present paper which is connected with previous ones of the author [Vestnik Moskov. Univ. **1950**, no. 10, 19–26; **1951**, nos. 3 and 6; **1953**, no. 9, 49–55 (see MR **14**, 251 for the first one)], aims at estimations of the solutions of (1) and at new necessary and sufficient conditions for stability of the same type.

If $G(x)=\sum a_{ih}x_ix_h$, $a_{ih}=a_{hi}$, $i, h=1, 2, \cdots, n$, is a definite positive real symmetric quadratic form, then the points $x=(x_1, \cdots, x_n)$ of the Euclidean x-space E where $G=c^2$ verify the relations $|x_s|\leqq|c|(A_{n-1}^{(s)}/A_n^0)^{1/2}$, $s=1, \cdots, n$, where A_r^s are the usual minors of the matrix $A=\|a_{ih}\|$ and A_n^0 is the determinant of A (§1). This is the main algebraic tool used by the author.

The author, in his first paper cited above, had already considered a particular quadratic form $G(t, x)=\sum a_{ih}(t)x_ix_h$, associated with system (1), having continuously differentiable coefficients, and obtained by a method of successive approximations. Denote by $G'=g(t, x)$ the derivative of G with respect to t relatively to (1). Denote by $G(t;\varphi(t, t_0, x^0))$ and $g(t, \varphi(t, t_0, x^0))$ the functions of t which are obtained by G and g when x is replaced by the solution $\varphi(t, t_0, x^0)$ of (1) taking the initial values $x=x^0$ as $t=t_0$. Let us also put $N_G(t)=\max g(t, x)$, $n_G(t)=\min g(t, x)$ for all $x\in E$ with $G(t, x)=1$, provided G is definite positive. Then the following evaluation of the solutions of (1) is obtained (§3):

$$|x_s(t)|\leqq\left\{G(t_0, x^0)(A_{n-1}^{(s)}(t)/A_n^0(t)\right.$$
$$\left.\times\exp\int_{t_0}^t g(\tau, x(\tau, t_0, x^0))d\tau\right\}^{1/2},\quad s=1, 2, \cdots, n.$$

This is the basic estimate of the solutions $x(t)$ of (1) used by the author.

The function G can be considered as a function V of Lyapunov and the author proves (§5) that a necessary and sufficient condition in order that the solution zero is stable for (1) is that a positive definite quadratic form $G(t, x)$ exists whose derivative $g(t, x)$ is nonpositive. A condition for asymptotic stability of the zero solution of (1) is that, in addition, the integral $\int_{t_0}^{+\infty}N_G(t)\,dt$ be divergent.

Given any norm N in E then the solution zero of (1) is said to be monotonically stable with respect to the norm N provided for every sufficiently small $\epsilon>0$ the solutions of (1) which are in the sphere with center the origin and radius ϵ (sphere with respect to the norm N) at $t=t_0$ remain there for all $t\geqq t_0$. Necessary and sufficient conditions for monotonic stability of the solution zero of (1), analogous to the ones above, are given. [A pertinent reference: J. L. Massera, Ann. of Math. (2) **50**, 705–721 (1949); MR **11**, 721.]
L. Cesari (Lafayette, Ind.).

Borůvka, Otakar. Remark on the use of Weyr's theory of matrices for the integration of systems of linear differential equations with constant coefficients. Časopis Pěst. Mat. **79**, 151–155 (1954). (Czech)

The author gives an explicit solution of the linear homogeneous system $y'=Ay$ with constant coefficients in terms of Weyr's normal system of vectors belonging to the matrix A.
M. Golomb (Lafayette, Ind.).

Švec, Marko. On the problem of uniqueness of integrals of a system of linear differential equations. Mat.-Fyz. Sb. Slovensk. Akad. Vied Umení 2, 3–22 (1952). (Slovak. Russian summary)

Suppose the functions $a_{ik}(x)$ are continuous in the closed interval F if x is a real variable, and analytic in the closed region F if x is a complex variable. Let $x_1, x_2, \cdots, x_n$ be points lying in the closed convex set $E\subset F$, and put $A_{ik}=\max_{x\in E}|a_{ik}(x)|$. The main result of this paper is: If diam $E<\rho^{-1}$ where ρ is the largest of the non-negative characteristic values of the matrix (A_{ik}), then the multipoint value problem $y_i'=\sum_{k=1}^n a_{ik}y_k$, $y_i(x_i)=0$, has no nontrivial solution. Nonhomogeneous systems are also considered and a brief remark is made about extension of the results to nonlinear systems. *M. Golomb.*

Have

***Thomas, Johannes. Über gewisse lineare Differentialgleichungssysteme mit periodischen Koeffizienten.** Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 226–230. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

See Math. Nachr. 9, 197–200 (1953); MR 14, 981.

Rabinovič, Yu. L. Estimate of the type and order of exponential growth of solutions of linear differential equations. Moskov. Gos. Univ. Uč. Zap. 165, Mat. 7, 205–207 (1954). (Russian)

Let $w^{(n)}=p_1(z)w^{(n-1)}+\cdots+p_n(z)w$ be a given nth order linear homogeneous differential equation with coefficients continuous and uniformly bounded in a neighborhood of ∞. Then there are two constants r and $\mu>0$ such that

$$|d^kw/dz^k|\leqq A|z|^{(r-1)k}\exp(\mu+1)|z|^r \quad (k=0, 1, \cdots, n-1).$$

This inequality refines a previous one of A. M. Lyapunov. Expressions for r and μ are given. The constant A depends on the solution w. *L. Cesari* (Lafayette, Ind.).

Popov, Blagoj S. Formation des critériums de réductibilité des équations différentielles linéaires ayant des formes données à l'avance. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 5 (1952), no. 2, 68 pp. (1954). (Macedonian. French summary)

Reducibility as understood in this paper for a linear homogeneous differential equation of second order means that it can be written in the form (a) $(D+\chi)(\varphi D+\psi)y=0$, where $D=d/dx$ and φ, ψ, χ are single-valued functions of x. Following the procedure of D. S. Mitrinovitch [C. R. Acad. Sci. Paris 230, 1130–1132 (1950); MR 11, 595] the author determines several classes of equations of the form

$$y''+(\alpha f+\beta)y'+(Af^2+Bf+C)y=0,$$

where α, β, A, B, C are constants and f is a function of x, which can be written in the form (a) with φ, ψ, χ specialized as (b) $\varphi=f^n+a_1f^{n-1}+\cdots+a_n$, $\psi=b_0f^{n+1}+b_1f^n+\cdots+b_{n+1}$, $\chi=c_0f+c_1$ (n an arbitrary positive integer). Explicit results are obtained for the cases $f=x$, $f=x^{-1}$, $f=e^x$, and $f=\tan x$, and these are applied to produce reducible cases of many of the classical equations. In each case the conditions are stated as necessary and sufficient for reducibility, whereas they are proved necessary only for reducibility as restricted by condition (b). *M. Golomb* (Lafayette, Ind.).

Latyševa, K. Ya. Normal solutions of linear differential equations with polynomial coefficients. Uspehi Mat. Nauk (N.S.) 8, no. 5(57), 205–212 (1953). (Russian)

This is a report by the author of her doctoral thesis dealing with normal and subnormal series in descending powers of x satisfying formally a linear differential equation of order n with polynomial coefficients. Conditions involving the coefficients of the equation are given, both necessary and sufficient or only sufficient for the existence of formal series solutions of the form

$$e^{Q(x)}x^r\sum c_ix^{-i},\quad e^{Q(x^{1/p})}x^{r/p}\sum c_ix^{-i/p},$$
$$e^{Q(x)}x^r\log^q x\sum c_ix^{-i},\quad e^{Q(x)}x^rS(x),$$

where Q is a polynomial, S a rational function, p and q positive integers. *M. Golomb* (Lafayette, Ind.).

Latyševá, K. Ya. On a result of B. S. Popov. Dokl. Akad. Nauk SSSR (N.S.) 96, 695–696 (1954). (Russian)

The reducible cases of the equation

$$y''+(ae^x+b)y'+(Ae^{2x}+Be^x+C)y=0$$

obtained by B. S. Popov [see the second preceding review] are derived anew by the author in the following way. With the substitution $x=e^t$ she obtains an equation with polynomial coefficients to which she applies her criterion for the existence of a solution of the form $y=e^{\alpha t}t^\rho P(t)$, where P is a polynomial [see the preceding review]. *M. Golomb.*

Latiševa, K. Ya. Subnormal series as solutions of linear differential equations whose rank equals unity. Dopovidi Akad. Nauk Ukrain. RSR 1952, 100–105 (1952). (Ukrainian. Russian summary)

To determine normal and subnormal series formally satisfying the equation (a) $y^{(n)}+\sum_{i=1}^n P_i(x)y^{(n-i)}=0$, where the $P_i(x)$ are regular at $x=\infty$, put $y=e^{\alpha x}u$ and u must satisfy the equation

$$u^{(n)}+\sum_{i=1}^n\frac{1}{(n-i)!}\frac{\partial^{n-i}}{\partial\alpha^{n-i}}Tu^{(n-i)}=0,$$

where $T(x,\alpha)=\sum_{i=0}^n b_i(\alpha)x^{-i}$. Assume the characteristic equation $b_0=0$ has the root α of multiplicity $q>1$. The author proves that if $b_i^{(\nu)}(\alpha)=0$ for $i=0, 1, \cdots, k-1$; $\nu=0, 1, \cdots, q-i-1$ and for $i=k, \cdots, s-1$; $\nu=0, 1, \cdots, s-i-1$, whereas $b_k^{(s-k)}(\alpha)\neq0$ $(k<s\leqq q)$, then there exist $s-k$ normal series $e^{\alpha x}x^r\sum c_ix^{-i}$ and $p=q-s+k$ subnormal series $\exp\{\alpha x+Q(x^{1/p})\}x^{r/p}\sum c_ix^{-i/p}$ with Q a polynomial of degree $\leqq q-s$. *M. Golomb* (Lafayette, Ind.).

Švec, Marko. Über einige neue Eigenschaften der (oscillatorischen) Lösungen der linearen homogenen Differentialgleichung vierter Ordnung. Čehoslovack. Mat. Ž. 4(79), 75–94 (1954). (Russian summary)

The linear homogeneous differential equations of order four considered in this paper are assumed to have only oscillatory solutions and the property (E) that no nontrivial solution has more than one zero of order $\geqq2$. For example, the equation (a) $y^{(4)}+Q(x)y=0$, where $Q(x)$ is continuous and positive, has property (E). The author proves several separation theorems for the zeros and extrema of the integrals of such equations. Typical samples: If $u(x)$, $\bar u(x)$ are two linearly independent integrals both of which have x_1 as a double zero then the simple zeros of $u(x)$, $\bar u(x)$ separate each other (Theorem 7). If $u(x)$, $\bar u(x)$ are two linearly independent integrals of (a) both of which have x_1 as a double zero then the zeros of $u'(x)$, $\bar u'(x)$ (other than x_1), also of $u''(x)$, $\bar u''(x)$ and of $u'''(x)$, $\bar u'''(x)$ separate each other (Theorem 11). *M. Golomb* (Lafayette, Ind.).

Bilharz, Herbert, und Schottlaender, Stefan. Periodische Lösungen einer geregelten Bewegung. Arch. Math. 5, 479–491 (1954).

The regulated motion under consideration is described by the equation (a) $\ddot x+2\zeta\dot x+x=-g$, where the controlling

force $g=\alpha x+\beta\dot{x}$ when $|\dot{g}|=|\alpha\dot{x}+\beta\ddot{x}|\leq V_0$ and takes on the saturation value $g=K_0\pm V_0t$ when $\dot{g}\gtrless V_0$, K_0 being determined by the condition that g be continuous. The authors determine all the periodic solutions of equation (a) for the various values of the regulating parameters α, β. For example, if there are three regions in the $\alpha\beta$-plane in which (a) has exactly 0, 1 and 2 periodic solutions respectively.

M. Golomb (Lafayette, Ind.).

Gusarov, L. A. On some properties of solutions of a linear differential equation of second order. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 223–237 (1954). (Russian)

The author gives a detailed analysis of some connections between bounds on the coefficient $p(x)$ in the equation $y''+p(x)y=0$ and distance between consecutive zeroes of the solution, the growth of the derivatives $y'(x)$ at these points, and the boundedness of the solution. A particularly interesting result is the theorem that $p(x)\geq a^2>0$, and $p'(x)$ continuous and of bounded variation for $x\geq x_0$ implies boundedness of all solutions of the above equation.

R. Bellman (Santa Monica, Calif.).

***Zachrisson, Lars Erik. On the energy levels of a generalized pendulum equation.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 324–337 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author considers a perturbed pendulum equation 1′) $\ddot{\varphi}+\sin\varphi=t^{-1}(\mu-\frac{1}{2}\dot{\varphi})$, constant $\mu\geq 0$, $t>0$, which is asymptotically like 1) $\ddot{\varphi}+\sin\varphi=0$. The solution-curve family on the cylinder can be projected onto the sphere S^2 and there 1) displays three centers with separatrices between them. As $t\to\infty$ each solution of 1′) on S^2 approaches a center or one of the separatrices of 1) on S^2. The author describes, without detailed analysis of the possible pathological complications, curve families on S^2 which are solutions of a differential system H like 1). Then a perturbed system S (the relations between S and H are too long to be given here) has solutions which are asymptotic to certain distinguished solutions (including centers, separatrices, and others) of H.

L. Markus (New Haven, Conn.).

Furuya, Shigeru. Van der Pol's equation with harmonic disturbance. Comment. Math. Univ. St. Paul. **3**, 7–13 (1954).

The author applies the theory of Kryloff and Bogoliuboff to the equation $\ddot{x}+\epsilon(x^2-1)\dot{x}+x=E\cos\omega t$ for the case $\omega=3(1+\epsilon\sigma)$ with ϵ a small parameter. If

$$\rho\cos(\tfrac{1}{3}\omega t+\phi)=x-\alpha\cos\omega t\quad(\alpha=E/(1-\omega^2)),$$

then in the first approximation ρ and ϕ satisfy

$$8\dot{\rho}=\epsilon\rho(4-2\alpha^2-\rho^2-\alpha\rho\cos 3\phi),\quad 8\dot{\phi}=-\epsilon(8\sigma-\alpha\rho\sin 3\phi).$$

The nature of the singular points of this system is examined and conditions on the parameters α and σ under which the system has a limit cycle are determined. The third-order subharmonic solutions of the Van der Pol equation have been treated earlier by H. G. Cohen [Publ. Sci. Tech. Ministère de l'Air, Paris no. **281**, 169–187 (1953); MR **15**, 313].

C. E. Langenhop (Ames, Iowa).

Yamaguti, Masaya. On some properties of the non-linear differential equations of the "parametric excitation." Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 87–96 (1954).

Conditions are given ensuring the ultimate boundedness of all solutions, and the existence of a periodic solution, of the equation (E): $\ddot{x}+f(x)\dot{x}+g(x,t)=p(t)$ with periodic $p(t)$. These conditions are satisfied when $f(x)=b>0$ and either (i) $g(x,t)=(\alpha+\beta\cos 2\omega t)x+\gamma x^3$, $\gamma>0$, and

$$p(t)=A\cos(\omega t+\phi),$$

(ii) $g(x,t)=(a-\epsilon x)x\cos 2t+ex^3$ and $p(t)=0$, where $e>0$, $\epsilon>0$, $\epsilon<2be/(1+2b)$. These appear to be the first results known for equation (E) when the restoring term g involves t and the non-linear terms are not assumed to be small.

G. E. H. Reuter (Manchester).

Malgarini, Giorgio. Studio asintotico del moto d'un oscillatore elastico, con resistenza di tipo "subviscoso." Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 258–280 (1953).

The differential equation considered is written in the form $\ddot{x}+H(\dot{x})+x=0$, whereby it is assumed that

$$\lim_{v\to 0} v^{-1}H(v)=+\infty,$$

that $vH(v)>0$ for $v\neq 0$, that $H(0)=0$, that $H(v)$ is continuous for all real v, that $H'(v)$ and $H''(v)$ exist for $v\neq 0$, that $H'(v)>0$, and that $vH''(v)<0$. It follows from these assumptions that $\lim_{v\to+\infty} v^{-1}H(v)=k_1$ and $\lim_{v\to-\infty} v^{-1}H(v)=k_2$ both exist and are non-negative. It is then proved that if at least one of the quantities k_1 and k_2 is greater than 2, the system can have no oscillations. If, however, both k_1 and k_2 are less than 2, it is possible to choose initial values for x and $\dot{x}$ such that the resulting solution $x(t)$ vanishes precisely n times as $t\to+\infty$. Here n is preassigned. The methods are largely of geometrical type.

D. C. Lewis.

Antosiewicz, H. A. On non-linear differential equations of the second order with integrable forcing term. J. London Math. Soc. **30**, 64–67 (1955).

For the differential equation (*) $\ddot{x}+\phi(x,\dot{x})\dot{x}+h(x)=e(t)$ the author shows that if $\phi(x,\dot{x})\geq 0$ for all x, $\dot{x}$,

$$H(x)=\int_0^x h(u)du>0$$

for all $x\neq 0$, $H(x)\to\infty$ with $|x|$, and $\int^\infty|e(t)|dt<\infty$, then every solution satisfies $|x(t)|<c_1$, $|\dot{x}(t)|<c_2$ as $t\to\infty$. (From the method of proof it would appear that c_1 and c_2 depend on the initial conditions of the solution.) In the case $\phi(x,\dot{x})=f(x)+g(x)\dot{x}$ the author establishes a similar result under the conditions $a(x)\leq\alpha$ for all x, $b(x)\cdot h(x)\geq 0$ and $H^*(x)=\int_0^x a^2(u)h(u)du>0$ for all $x\neq 0$, $H^*(x)\to\infty$ with $|x|$, and $\int^\infty|e(t)|dt<\infty$, where

$$a(x)=\exp\left(\int_0^x g(u)du\right),\quad b(x)=\int_0^x a(u)f(u)du.$$

The author observes that these two together give alternative sufficient conditions for boundedness of the solutions of (*) in the case $\phi(x,\dot{x})=f(x)$.

C. E. Langenhop.

Colombo, Giuseppe. Sulle oscillazioni forzate di un circuito comprendente una bobina a nucleo di ferro. II. Rend. Sem. Mat. Univ. Padova **23**, 407–421 (1954).

This is a reconsideration of a problem discussed in a previous paper with the same title by the author [same Rend. **22**, 380–398 (1953); MR **15**, 427].

C. R. De Prima (Pasadena, Calif.).

Minorsky, N. Sur la méthode stroboscopique. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **9**, 23–29 (1952).

Summary of the method previously described in Bull. Soc. Franç. Méc. **4**, no. 13, 15–26 (1954); MR **16**, 131.

A. S. Householder (Oak Ridge, Tenn.).

Aymerich, G. Modulazione di ampiezza e di fase nell'oscillatore di Rocard. Rem. Sem. Fac. Sci. Univ. Cagliari 23, 165–174 (1953).

Continuing an investigation begun in a paper in same Rend. 22, 109–116 (1953) [MR 15, 707], the author proves the existence of combination oscillations for Rocard's oscillator. Mathematically the problem consists in establishing the existence of stable periodic solutions for the system

$$\frac{d\xi}{d\tau}=\frac{\xi}{(\xi^2+\eta^2)^{1/2}}-(\xi+\sigma\eta),\quad \frac{d\eta}{d\tau}=\frac{\eta}{(\xi^2+\eta^2)^{1/2}}-F+(\sigma+\xi-\eta),$$

when the constants F and σ satisfy the inequalities $F>1$, $\sigma>4F^2-1$. The proof consists in an application of Bendixson's theorem, which is possible, as the author shows, in spite of the discontinuity at $\xi=\eta=0$. *W. Wasow.*

Sansone, G., e Conti, R. Sull'equazione di T. Uno ed R. Yokomi. Ann. Mat. Pura Appl. (4) 37, 37–59 (1954).

The authors give a careful discussion of the characteristics of the system of equations

$$\dot{x}=y^2-(x+1)[(x-1)^2+\lambda],\quad \dot{y}=-xy,$$

where λ is a real parameter. In particular, they investigate the existence, stability, and location of limit cycles. It is found that if $\lambda\leqq 0$ or $\lambda\geqq 1$, there are no limit cycles. If $0<\lambda<1$, there exist just two limit cycles, which are stable and which are obtainable, one from the other, by a reflection in the x-axis. If $\lambda=0$, the point $(-1, 0)$ is a saddle-point, the point $(+1, 0)$ is a singular point of a higher type, and there is a separatrix α proceeding from the first of these points to the second. For small positive values of λ, the limit cycle in the upper half-plane is represented approximately by the cycle composed of α and the segment of the x-axis joining the points $(\pm 1, 0)$. For values of λ such that $1-\lambda$ is positive and sufficiently small, the limit cycle in the upper half-plane lies in an arbitrarily small neighborhood of the point $(0, \sqrt{2})$.

The progressive change of the family of characteristics as λ varies over the range $5-4\sqrt{2}<\lambda<5+4\sqrt{2}$ is studied in some detail. It is pointed out that one implication of the results is that the generalized Liénard equation

$$\ddot{x}+[3x^2-(1-\lambda)]\dot{x}+2x(x+1)[(x-1)^2+\lambda]=0$$

has no periodic solution if $\lambda\leqq 0$ or $\lambda\geqq 1$, whereas if $0<\lambda<1$, the equation has a unique and stable periodic solution.

L. A. MacColl (New York, N. Y.).

Greguš, Michal. Application of dispersions to boundary problems of the second order. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 27–37 (1954). (Slovak. Russian summary)

Let a and $\varphi(a, \lambda)>a$ be two consecutive zeros of an integral of the equation

$$\text{(a)}\qquad [\theta(x, \lambda)y']'-Q(x, \lambda)y=0.$$

$\varphi(x, \lambda)$ is the "dispersion (of the first kind) of order 1" of equation (a) and was introduced by Borůvka [Čehoslovack. Mat. Ž. 3(78), 199–255 (1953); MR 15, 706] for the equation $y''+Q(x)y=0$. It is shown to satisfy the equation

$$\text{(b)}\qquad \varphi'(x, \lambda)=\rho^2(\varphi, \lambda)\theta(\varphi, \lambda)/\rho^2(x, \lambda)\theta(x, \lambda),$$

where $\rho^2=z_1^2+z_2^2$ and z_1, z_2 are two linearly independent integrals of (a). By using (b) and other properties of the dispersion the author proves Sturm's oscillation theorems for the integrals of equation (a). *M. Golomb.*

Hartman, Philip. On the essential spectra of ordinary differential operators. Amer. J. Math. 76, 831–838 (1954).

Let $q(t)$, $0\leqq t<\infty$, be continuous and tend monotonously to ∞ with t. The author considers what additional condition will ensure that the essential spectrum S' of $y''+(\lambda+q)y=0$ is the entire real λ-axis; he has previously shown the necessity of such a condition [same J. 74, 107–126 (1952), appendix; MR 14, 558]. One such condition is $q(t)=o(t^2)$ as $t\to\infty$, and another is $\int^\infty q^{-\frac{1}{2}\mu}dt=\infty$ for some $\mu>1$ (this can be slightly weakened); the proof rests on an oscillatory characterisation of S'. Passing to the case of $q(t)$ bounded from above, a result of C. R. Putnam [ibid. 74, 578–586 (1952); MR 14, 472] on the order of magnitude of gaps in S' is re-proved, and shown by an example to be precise.

F. V. Atkinson (Ibadan).

Grohne, D. Über das Spektrum bei Eigenschwingungen ebener Laminarströmungen. Z. Angew. Math. Mech. 34, 344–357 (1954). (English, French and Russian summaries)

The study of small disturbances of plane parallel viscous flows leads to the differential equation

$$(U(y)-c)(\phi''(y)-\alpha^2\phi(y))-U''(y)\phi(y)$$
$$=\frac{1}{i\alpha R}(\phi^{(4)}(y)-2\alpha^2\phi''(\alpha)+\alpha^4\phi(y))$$

for the function ϕ and to the boundary conditions $\phi(\pm 1)=\phi'(\pm 1)=0$. The author investigates the spectrum of the eigenvalue parameter c in dependence on the positive parameters α, R. For $U(y)\equiv 0$ the problem has a discrete spectrum $c=C_n$ $(n=1, 2, \cdots)$, which can be easily calculated [Rayleigh, Sci. Papers, v. III, Cambridge, 1902, pp. 575–584]. The author shows that the general problem possesses eigenvalues of the form $c_n=C_n+\frac{1}{2}\int_{-1}^{1}U\,dy+O(n^{-1})$. The case $U(y)=y$ is studied asymptotically, as $\alpha R\to\infty$, by means of Bessel functions, and by numerical methods for moderate αR. This part of the paper has much in common with an article by the reviewer [J. Res. Nat. Bur. Standards 51, 195–202 (1953); MR 15, 573]. When $U(y)$ is symmetric the author shows that there exists a family of eigenvalues that approach asymptotically, as $\alpha R\to\infty$, those for the case $U(y)=y$. He also gives an expression for the second asymptotic term. Another type of eigenvalues, that tend to those of the corresponding inviscid problem, are discussed more briefly. Finally, the author indicates how his results can be used to simplify the search for eigenvalues that correspond to unstable disturbances. A third class of eigenvalues, always strongly damped, whose existence was proved by C. Morawetz [J. Rational Mech. Anal. 1, 579–603 (1952); MR 14, 509] is not included in the discussion.

W. Wasow (Rome).

Partial Differential Equations

Have

***Petrovsky, I. G. Lectures on partial differential equations.** Translated by A. Shenitzer. Interscience Publishers, New York-London, 1954. x+245 pp. $5.75.

Translation of the author's Lekcii ob uravneniah c častnymi proizvodnymi [Gostehizdat, Moscow, 1950; MR 13, 241].

Rachajsky, B. Sur les transformations de contact. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 896–909 (1954).

Verfasser behandelt die folgenden Probleme: (1) liegt eine Berührungstransformation vor, so sollen alle partiellen Differentialgleichungen erster Ordnung für eine unbekannte Funktion und n unabhängige Veränderliche aufgestellt werden, die sich durch die gegebene Berührungstransformation auf eine Funktionalgleichung reduzieren; (2) liegt umgekehrt eine partielle Differentialgleichung erster Ordnung für eine unbekannte Funktion und n unabhängige Veränderliche vor, so soll (wenigstens) eine Berührungstransformation gewonnen werden, welche die gegebene Differentialgleichung in eine Funktionalgleichung überführt; (3) das für (1) bzw. (2) gestellte Problem soll auf den Fall eines Involutionssystems $\Phi_i(x_1, \cdots, x_n, p_1, \cdots, p_n)=0$, $i=1, \cdots, m<n$, verallgemeinert und gelöst werden, Bezüglich (1) und (2) sind die Ausführungen des Verfassers Verallgemeinerungen einer früheren Arbeit des Verfassers von zwei unabhängigen Veränderlichen auf deren n und es kann auf die Besprechung dieser Arbeit verwiesen werden [Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 79–90 (1953); MR 16, 252]. Das Problem (3) wird allgemein diskutiert und die Ergebnisse werden auf ein von Günther angegebenes Beispiel eines Involutionssystems praktisch angewendet. *M. Pinl* (Köln).

Aržanyh, I. S. On a method of integration of partial differential equations of the first order. Uspehi Mat. Nauk (N.S.) 9, no. 3(61), 115–118 (1954). (Russian)

Verfasser ersetzt die partielle Differentialgleichung

$$(*)\quad F(z, x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n)=0, \quad p_\nu=\partial z/\partial x_\nu,$$

durch das System

$$(**)\quad \begin{gathered} H(z, x_1, x_2, \cdots, x_n, q_1, q_2, \cdots, q_n)=0, \\ p_\nu=Q_\nu(z, x_1, \cdots, x_n, q_1, \cdots, q_n), \end{gathered}$$

d.h. er nimmt an, dass (*) aus (**) durch Elimination der $q_1, \cdots, q_n$, entsteht. Sodann gilt: sind

$$(***)\quad \varphi_\nu(z, x_1, x_2, \cdots, x_n, q_1, q_2, \cdots, q_n, c_1, c_2, \cdots, c_n)=0 \quad (\nu=1, 2, \cdots, n)$$

die Integrale von

$$\frac{dx_\nu}{dt}=\frac{\partial H}{\partial q_\nu}, \quad \frac{dq_\nu}{dt}=-\frac{\partial H}{\partial x_\nu}-\frac{\partial H}{\partial z}Q_\nu, \quad \frac{dz}{dt}=\sum_{\mu=1}^{n}\frac{\partial H}{\partial q_\mu}Q_\mu,$$

wobei die Funktionen φ_ν den Involutionsbedingungen

$$\sum_{\lambda=1}^{n}\left[\frac{\partial\varphi_\nu}{\partial q_\lambda}\left(\frac{\partial\varphi_\mu}{\partial x_\lambda}+\frac{\partial\varphi_\mu}{\partial z}Q_\lambda\right)-\frac{\partial\varphi_\mu}{\partial q_\lambda}\left(\frac{\partial\varphi_\nu}{\partial x_\lambda}+\frac{\partial\varphi_\nu}{\partial z}Q_\lambda\right)\right]=0,$$

$$\frac{\partial(\varphi_1\varphi_2\cdots\varphi_n)}{\partial(q_1q_2\cdots q_n)}\neq 0,$$

genügen, so ergibt sich ein vollständiges Integral der Gleichung (*) durch Elimination der $q_1, \cdots, q_n$ aus den Integralen (***) und der Beziehung $H=0$. Als Beispiel zu diesem Ergebnis wird die Clairautsche Gleichung

$$z=\sum_{\nu=1}^{n}x_\nu p_\nu+f(p_1, p_2, \cdots, p_n)$$

behandelt. Die entwickelte Methode liefert auch die hinreichende Anzahl von Bedingungen, die notwending sind, um das Cauchysche Problem der Gleichung (*) zu lösen. *M. Pinl* (Köln).

Aržanyh, I. S. Extension of the method of characteristics to simultaneous partial differential equations of the first order. Uspehi Mat. Nauk (N.S.) 9, no. 3(61), 119–125 (1954). (Russian)

An Stelle einer einzelnen partiellen Differentialgleichung erster Ordnung in einer abhängigen und n unabhängigen Veränderlichen [vgl. die oben referierte Arbeit] betrachtet Verfasser nunmehr ein Involutionssystem

$$F_\rho(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=0, \quad p_\nu=\partial z/\partial x_\nu, \quad \rho=1, 2, \cdots, r, \quad \nu=1, 2, \cdots, n.$$

Zur Lösung des Cauchyschen Problems hat man durch eine vorgegebene Anfangsmannigfaltigkeit

$$\pi_\rho(x_1, \cdots, x_n)=0 \quad (\rho=1, 2, \cdots, n), \quad z=\pi(x_1, \cdots, x_n)$$

eine Integralmannigfaltigkeit hindurchzulegen. Dies geschieht im Falle $r=1$ durch die charakteristischen Kurven, welche die Integralfläche erzeugen (Cauchy's Integrationsmethode). Im vorliegenden allgemeinen Falle sind die charakteristischen Kurven durch charakteristische Mannigfaltigkeiten

$$\omega_i(x_1, \cdots, x_n)=0 \quad (i=1, \cdots, n-r), \quad z=\omega(x_1, \cdots, x_n)$$

zu ersetzen, um die Integralmannigfaltigkeit zu erzeugen. Zur Entwicklung der Methode müssen zunächst Anfangsmannigfaltigkeit wie auch charakteristische Mannigfaltigkeit auf Parameter $s_1, \cdots, s_{n-r}$ bzw. $t_1, \cdots, t_r$ bezogen werden. Ferner müssen die Bedingungen aufgestellt werden, welche die Verträglichkeit der Anfangsbedingungen mit den Gleichungen des Involutionssystems zum Ausdruck bringen. Sodann gilt: die charakteristischen Mannigfaltigkeiten werden durch die Gleichungen

$$(*)\quad dx_\nu=\sum_\rho\frac{\partial F_\rho}{\partial p_\nu}dt_\rho, \quad -dp_\nu=\sum_\rho\left(\frac{\partial F_\rho}{\partial x_\nu}+p_\nu\frac{\partial F_\rho}{\partial z}\right)dt_\rho,$$

$$dz=\sum_\nu\sum_\rho p_\nu\frac{\partial F_\rho}{\partial p_\nu}dt_\rho$$

bestimmt. Ferner: dieses System (*) ist zufolge des Involutionssystems $F_\rho=0$ vollständig integrabel. Schliesslich: die Integrale

$$\begin{gathered}\Omega(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=c, \\ \Omega_\nu(x_1, \cdots, x_n, z, p_1, \cdots, p_n)=c_\nu\end{gathered}$$

sind so beschaffen, dass die Ungleichung

$$\partial(\Omega, \Omega_1, \cdots, \Omega_n)/\partial(z, p_1, \cdots, p_n)\neq 0$$

mit dem Involutionssystem verträglich ist. Auf Grund dieser Ergebnisse kann nunmehr das Cauchysche Problem für das gegebene Involutionssystem gelöst werden. Das Verfahren vereinfacht sich erheblich, wenn die unbekannte Funktion z im Involutionssystem nicht explizit enthalten ist. Als Beispiel wird das System

$$p_1p_2p_3=\frac{1}{x_1x_2x_3}, \quad p_1x_1+p_2x_2+p_3x_3=0$$

behandelt. *M. Pinl* (Köln).

Cinquini-Cibrario, Maria. Sopra la teoria delle caratteristiche per i sistemi di equazioni non lineari alle derivate parziali del primo ordine. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 17(86), 725–746 (1953).

In the system (E) $F_i(x, y; z_j; z_{jx}, z_{jy})=0$ $(i, j=1, \cdots, n)$ the functions F_i are assumed to have Lipschitz-continuous third (or, in later theorems, second) partial derivatives in a region D of $(3n+2)$-space. With

$$P_{ij}=\partial F_i/\partial p_j \text{ and } Q_{ij}=\partial F_i/\partial q_j \quad (p_j=z_{jx}, q_j=z_{jy}),$$

it is assumed at the outset that the roots of

$$\text{(C)} \qquad |P_{ij}\rho - Q_{ij}| = 0$$

are real, distinct, finite, and $\neq 0$. It is stated what is meant by integral surface and by characteristic strip relative to a root ρ of (C), and what it means for an integral surface to contain a characteristic strip. The system (E) is first transformed to a quasi-linear system (Q) with $3n$ unknowns (z_j, p_j, q_j). Three theorems on the characteristic initial-value problem are then obtained by applying to (Q) the results of an earlier study [Ann. Scuola Norm. Super. Pisa (3) **3**, 161–197 (1950); MR **12**, 337]. Results are also obtained on the first-order characteristic strips of (E). Then conditions are given under which a certain Cauchy problem either is inconsistent or has infinitely many solutions. Further results concern characteristic strips of higher order and the situation arising when (C) has multiple roots.

F. A. Ficken (Knoxville, Tenn.).

Hornich, Hans. Das Problem der linearen Differentialoperatoren. Rend. Sem. Mat. Univ. Padova **23**, 333–339 (1954).

Let Lu be the linear differential operator

$$Lu = \sum g_{i_1\cdots i_n}(x_1, \cdots, x_n)\frac{\partial^i u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} \qquad (0 \leqq i = i_1 + \cdots + i_n \leqq r),$$

where the coefficients are continuous in a neighborhood U of a point P. The equation $Lu = f$ is said to be solvable for all f if there is a neighborhood $U' \subset U$ in which, for every continuous f, there exists a solution of $Lu = f$ which is a continuous function and has continuous derivatives of every order corresponding to a non-vanishing coefficient $g_{i_1\cdots i_n}$. The author first proves a theorem which gives a condition on the coefficients which is sufficient to make $Lu = f$ not solvable for all f. He then proves the theorem that if some $g_{r_1\cdots r_n}(x)$ is not 0 at P, and if all coefficients $g_{i_1\cdots i_n}$ which do not vanish identically have $i_1 \leqq r_1, \cdots, i_n \leqq r_n$, then $Lu = f$ is solvable for all f. For the case $n = 1$, when the equation becomes an ordinary differential equation of order r, these 2 theorems give a necessary and sufficient condition that $Lu = f$ be solvable for all f: that the coefficient of the derivative of order r, $g_r(x)$, does not vanish at P.

D. L. Bernstein (Rochester, N. Y.).

Višik, M. I. On the first boundary problem for elliptic differential equations with operator coefficients. Soobšč. Akad. Nauk Gruzin. SSR **13**, 129–136 (1952). (Russian)

Let D be a bounded domain of Euclidean n-space, and let A_{jk}, B_j, C_j, F $(j, k = 1, \cdots, n)$ be bounded operators on $L^2(D)$. Suppose that: (1) for an arbitrary $v^0 \in \Omega^0(D)$, the set of complex-valued infinitely differentiable functions with compact support in D, $A_{jk}v^0$, C_jv^0 are continuously differentiable in D with respect to x_j; $A_{jk}v^0$, B_kv^0 are continuously differentiable in D with respect to x_k; all these derivatives lie in $L^2(D)$; (2) there exists $\mu > 0$ such that for $u^0 \in \Omega^0(D)$,

$$\sum_{j,k=1}^{n}\left[(A_{jk} + A_{kj})\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_j}\right] \geqq \mu \sum_{j=1}^{n}\left[\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j}\right].$$

Using suitable weak extensions of the operators involved, the author considers solutions of the Dirichlet problem (in a generalized sense) for the equation

$$Lu = -\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}\left(A_{jk}\frac{\partial u}{\partial x_k}\right) + \sum_{j=1}^{n}\left(B_j\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}C_j\right)u + Fu = f.$$

The Fredholm alternative is established for this equation and a suitably defined adjoint equation. *F. Browder.*

★**Loewner, C. Conservation laws of certain systems of partial differential equations and associated mappings.** Contributions to the theory of partial differential equations, pp. 161–165. Annals of Mathematics Studies, no. 33. Princeton University Press, Princeton, N. J., 1954. $4.00.

Summary of results in J. Rational Mech. Anal. **2**, 537–561 (1953); MR **15**, 73.

Mann, W. Robert, and Blackburn, Jacob F. A nonlinear steady state temperature problem. Proc. Amer. Math. Soc. **5**, 979–986 (1954).

The problem (P) is to solve $U_{xx} + U_{yy} = 0$ on a semi-infinite strip with (a) $U(0, y) = 0 = U(\pi, y)$ $(y \geqq 0)$, (b) $U_y(x, 0) = -G(U(x, 0))$ $(0 < x < \pi)$, and (c) $|U(x, y)| < M$ for $0 \leqq x \leqq \pi$, $0 \leqq y$, and some $M < \infty$. Here $G(U)$ is continuous, does not increase, and has $G(1) = 0$. By means of a finite Fourier transform the problem is reduced to the non-linear integral equation (E) $U(x, 0) = \int_0^\pi K(x, \lambda)G(U(\lambda, 0))\,d\lambda$ with

$$K(x, \lambda) = (2\pi)^{-1}\log\{[1 - \cos(x + \lambda)][1 - \cos(x - \lambda)]^{-1}\}.$$

By an argument involving a fixed-point theorem of Schauder's, it is shown that (E) has a unique bounded solution $U(x, 0)$, that $U(x, 0)$ is continuous, and that $0 \leqq U(x, 0) \leqq 1$. A formula is given for calculating a solution $U(x, y)$ of (P) from $U(x, 0)$, and it is stated that the validity of this formula can be verified by routine methods.

F. A. Ficken (Knoxville, Tenn.).

Horváth, J. I. Bemerkungen zur Lösung der Schrödinger-Gleichung mittels des Variationsverfahrens. Acta Phys. Acad. Sci. Hungar. **3**, 323–327 (1954).

Mihlin, S. G. Integration of Poisson's equation in an infinite region. Dokl. Akad. Nauk SSSR (N.S.) **91**, 1015–1017 (1953). (Russian)

Let Ω be an unbounded open set in m-dimensional Euclidean space, with a bounded boundary S, and consider the integration in Ω of Poisson's equation $-\Delta u = f(x)$; $f(x) \in L_2(\Omega)$, subject to the boundary condition $u|_S = 0$, this boundary condition being understood in the sense of Courant and Hilbert [Methoden der mathematischen Physik, vol. 2, Springer, Berlin, 1937] and S. L. Sobolev [Some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565]. The operator $-\Delta$, considered on the subset M of $L_2(\Omega)$ consisting of all functions which are twice continuously differentiable on Ω and vanish on a "boundary strip" of S, is positive, but not positive definite [for the terminology used see S. G. Mihlin, Direct methods in mathematical physics, Gostehizdat, Moscow-Leningrad, 1950; MR **16**, 41]. Now, if A is a positive operator with domain of definition $D(A)$ in a Hilbert space H and values in H, the solution of the equation $Au = f$, $f \in H$, is equivalent to the minimization of the functional $F(u) = (Au, u) - (u, f) - (f, u)$. If A is positive, but not positive definite, then the completion H_A of $D(A)$ with respect to the metric defined by $[u, v] = (Au, v)$, $|u|^2 = (Au, u)$ will contain "ideal elements" which can not be identified with elements of H. However, if the functional (u, f) (which is defined on the common part of H_A and H) is bounded in H_A then $(u, f) = [u, u_0]$ for some fixed, uniquely determined $u_0 \in H_A$. Hence $F(u) = |u - u_0|^2 - |u_0|^2$

for $u \in H_A$, and given $f \in H$, the element u_0 minimizes the functional $F(u)$ over H_A. If u_0 is in H_A, but not in H, then it is called a "singular" solution of $Au=f$; if u_0 is in H it is called a "generalized" solution. Put $A=-\Delta$; the present paper contains four theorems serving to characterize the "singular" solutions of Poisson's equation. *J. B. Diaz.*

Netanyahu, E. On the singularities of solutions of differential equations of the elliptic type. J. Rational Mech. Anal. **3**, 755–761 (1954).

This paper is concerned with elliptic partial differential equations $\Delta u+Au_x+Bu_y+C=0$, where A, B, C, are entire analytic functions of x and y (if the latter are regarded as complex variables). If (a) $U(z, z^*)=\sum_{m,n=0}^{\infty} D_{nm}z^m z^{*n}$, the author shows that the location and character of the singularities of $U(z, z^*)$ are determined by the sequence of partial sums of the expansion (a) which are obtained by setting $n=$const. The proofs are based on Bergman's integral operator method. The author also considers expansions of the more general type $\sum_{m,n=0}^{\infty} D_{nm}(z-a)^m(z^*-a^*)^n$.

Z. Nehari (Pittsburgh, Pa.).

Have

***de Schwarz, Maria Josepha. Su un sistema di equazioni differenziali a derivate parziali concernente gli spostamenti nelle volte cilindriche sottili.** Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 82–88. Casa Editrice Perrella, Roma, 1953.

The system of partial differential equations considered is the following:

$$(1)\qquad \begin{aligned} &\Delta u+\kappa\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+b\frac{\partial w}{\partial x}=X,\\ &\Delta v+\kappa\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+c\frac{\partial w}{\partial y}=Y,\\ &b\frac{\partial u}{\partial x}+c\frac{\partial v}{\partial y}+d\Delta\Delta w+aw=Z, \end{aligned}$$

where X, Y, Z are given functions, a, b, c, d, κ are constants, and $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$. Boundary conditions are specified on the boundary of the rectangle $0\leqq x\leqq x_1$; $0\leqq y\leqq y_1$. If $X=Y=Z=0$, then each of the functions u, v, w is a linear combination of the derivatives of a function F satisfying an equation with constant coefficients:

$$(2)\qquad \Delta^4 F+\frac{1-\nu^2}{\rho^2 k}\frac{\partial^4 F}{\partial x^4}=0.$$

The particular system (1) arising in applications is treated by seeking a complete system of solutions of the partial differential equation (2) which lends itself to numerical calculations, a method which has been developed by M. Picone and his collaborators. *J. B. Diaz.*

Have

***Diaz, J. B., and Weinstein, Alexander. On the fundamental solutions of a singular Beltrami operator.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 97–102. Academic Press Inc., New York, 1954. $9.00.

Let Δ_2 be Beltrami's operator in the case of the line element

$$ds^2=\frac{dx_1^2+\cdots+dx_m^2+dy^2}{y^{2k/(1-m)}},\quad m=2, 3, \cdots (k=\text{real constant}).$$

Then the equation $\Delta_2 u=0$ is reduced to the following form

$$(*)\qquad \frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_m^2}+\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=0.$$

By a generalization of Gauss' formula for the potential of a circular ring in the ordinary three-dimensional Euclidean space E_3 the authors get the fundamental solution

$$u_b^{(k)}=\int_0^{\pi}\frac{\sin^{k-1}\alpha\, d\alpha}{(\sum_{i=1}^{m} x_i^2+b^2+y^2-2by\cos\alpha)^{(k+m-1)/2}}$$

of the differential equation (*) in the sense of J. Hadamard [Ann. Sci. Ecole Norm. Sup. (3) **21**, 535–556 (1904); **22**, 101–141 (1905)]. Between the solutions $u^{(k)}$ the correspondence principle (**) $u^{(k)}=y^{1-k}u^{(2-k)}$ holds which is a link between solutions of (*) for the indices k and $2-k$ which remains valid also for $m=2, 3, \cdots$. Using the correspondence principle (**) and the formula (*), the authors get a second fundamental solution of the problem. *M. Pinl.*

Vasilache, Sergiu. Sur le problème de Neumann intérieur pour l'équation générale de type elliptique. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 231–241 (1952). (Romanian. Russian and French summaries)

The author considers the boundary-value problem

$$u_{xx}+u_{yy}+u_{zz}=\lambda[au_x+bu_y+cu_z+qu]+b,\ \text{in } D,$$

$$\frac{du(P)}{dn_1}=0,\quad n_1 \text{ the inner normal, on } S,$$

where D is a domain in three-dimensional Euclidean space and S is its (smooth) boundary surface; λ is a parameter and a, b, c, q, f are given functions, continuous on D. The problem is shown to be equivalent to a certain integro-differential equation involving a kernel $G(M, P)$, similar to Neumann's function, which has to satisfy several additional restrictions. In this paper the kernel $G(M, P)$ is determined; the integral equation itself is to be studied later.

J. B. Diaz (College Park, Md.).

Vasilache, Sergiu. Sur une classe d'équations intégro-différentielles que l'on rencontre dans la théorie des équations aux dérivées partielles. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 505–517 (1952). (Romanian. Russian and French summaries)

The boundary-value problem of the paper reviewed above leads to an integral equation which is a special case of

$$(**)\qquad u(M)=F(M,\lambda)+\lambda\int_D G(M,P)Lu(P)d\tau_P,\quad \text{for } M, P \text{ in } D,$$

where D is now a d-dimensional domain;

$$F(M,\lambda)=\sum_{i=0}^{n}\frac{f_i(M)}{\lambda^i}+\sum_{j=1}^{\infty}h_j(M)\lambda^j,$$

with f_i, for $i=1, \cdots, n$, and h_j, for $j=1, 2, 3, \cdots$, and $G(M, P)$, being given functions; and L is a linear, homogeneous differential operator. The integral equation (**) is solved by successive approximations, and it is shown how the results apply to the solution of the integral equation arising in the Neumann problem (*). *J. B. Diaz.*

Friedlander, F. G., and Keller, Joseph B. Asymptotic expansions of solutions of $(\nabla^2+k^2)u=0$. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **EM-67**, i+10 pp. (1954).

In many diffraction problems it is necessary to determine the asymptotic behaviour as $k\to\infty$ of solutions of

$$(*)\qquad \nabla^2u+k^2u=0.$$

If an exact solution of the problem is known, there is no difficulty. But if there is no known exact solution, some direct procedure is desirable.

Methods initiated by R. K. Luneberg do not yield all possible asymptotic solutions of electromagnetic diffraction problems, in that they cannot bring in fractional powers of k or exponential decay factors. In this paper, solutions of (*) of the form $u=v\exp(ik\phi-k^\alpha\psi)$ are considered where $v\sim\sum_0^\infty v_n(x,y,z)/k^{\lambda_n}$; where ϕ, ψ, v_n and the real numbers α, λ_n are to be determined in such a way that $\lambda_{n+1}>\lambda_n$. In the procedure developed here, ϕ may be any solution of the eikonal equation $(\text{grad }\phi)^2=1$. The surfaces $\psi=\text{constant}$ are ruled surfaces generated by the rays which are straight lines orthogonal to the surfaces $\phi=\text{constant}$. The constant α must lie in the range $0<\alpha\leq\frac{1}{2}$. If $\alpha=\frac{1}{2}$, $\lambda_n=\frac{1}{2}n$. But if $0<\alpha<\frac{1}{2}$, λ_n is the $(n+1)$th member of the sequence generated by arranging the numbers

$$m_1(1-2\alpha)+m_2(1-\alpha)+m_3$$

in increasing order of magnitude, m_1, m_2, m_3 being any non-negative integers. *E. T. Copson* (St. Andrews).

Have

★Pólya, G. Estimates for eigenvalues. Studies in mathematics and mechanics presented to Richard von Mises, pp. 200–207. Academic Press Inc., New York, 1954. $9.00.

The author considers the problem of finding upper bounds for the eigenvalues λ_n of a vibrating membrane with the aid of Poincaré's theorem: $\lambda_n=\min_{S_n}\max_{f\in S_n}R(f)$, where $R(f)$ is the Rayleigh ratio $-(u,\Delta u)/(u,u)$ [Amer. J. Math. 12, 211–294 (1890)]. By taking for S_n a specific linear space of dimension n, $\max_{f\in S_n}R(f)$ furnishes an upper bound for λ_n. The author contrasts this with the classical Courant principle: $\lambda_n=\max_{T_{n-1}}\min_{f\perp T_{n-1}}R(f)$. The first application is to estimating eigenvalues of domains D which are affinely similar to domains D' for which the eigenfunctions can be calculated explicitly. For such domains the author chooses for S_n the space spanned by the affine transpolation of the first n eigenfunctions in D'. The second application is to domains which are the union of a finite number of lattice squares. For these he proposes for S_n the space of all piecewise bilinear functions; n is equal to the number of interior lattice points. The eigenvalues of $R(f)$ over S_n are eigenvalues of the system of equations

$$(u_{i+1,j}+\cdots+u_{i+1,j+1}+\cdots-iu_{ij})/3h^2$$
$$=-(\lambda/36)(16u_{ij}+4u_{i+1,j}+\cdots+u_{i+1,j+1}+\cdots].$$

The eigenvalues of this system of equations are thus upper bounds for the true eigenvalues. In contrast, the lowest eigenvalue furnished by the standard finite-difference scheme furnishes a lower bound for the lowest eigenvalue, as G. E. Forsythe has shown recently, at least for a restricted class of domains [Pacific J. Math. 4, 467–480 (1954); MR 16, 179]. *P. D. Lax* (New York, N. Y.).

Saul'ev, V. K. Proof of convergence of the eigenfunctions obtained by the method of grids. Uspehi Mat. Nauk (N.S.) 9, no. 4(62), 217–224 (1954). (Russian)

Uniform convergence of eigenfunctions of all orders is shown for $n\leq5$ for the problem $\sum_1^n\partial^2u/\partial x_i^2+\lambda u=0$ in a region Q with u vanishing on the boundary.

A. S. Householder (Oak Ridge, Tenn.).

Levitan, B. M. On expansion in eigenfunctions of the equation $\Delta u+\{\lambda-q(x_1,x_2,\cdots,x_n)\}u=0$. Dokl. Akad. Nauk SSSR (N.S.) 97, 961–964 (1954). (Russian)

Let D be a finite simply-connected region of Euclidean n-space E_n $(n>3)$, and B its boundary. The eigenvalue problem $-\Delta u+qu=\lambda u$, $\partial u/\partial n=0$ on B, is considered where q is real and of class C^{n-3} on D. Results are stated which are analogous to those previously announced by the author for the case $n=3$ [same Dokl. (N.S.) 94, 179–182 (1954); MR 15, 797]. These involve asymptotic estimates for the spectral function, and convergence of the Riesz means of the expansion by eigenfunctions to the Riesz means associated with the ordinary Fourier expansion. It is indicated how these results carry over to the case when D is replaced by E_n, and when the given differential operator is generalized to

$$-\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+qu,$$

where $a_{ij}=a_{ji}$, and $a_{ij}\in C^{n-1}$. *E. A. Coddington.*

Bojanić, Ranko, und Vučković, Vladeta. Über die Eigenfunktionen der schwingenden Platte. Srpska Akad. Nauka. Zb. Rad. 35, Mat. Inst. 3, 107–128 (1953). (Serbo-Croatian. German summary)

Suppose $\lambda_1\leq\lambda_2\leq\cdots$ are the eigenvalues, $\phi_1(P)$, $\phi_2(P)$, $\cdots$ the corresponding orthonormal eigenfunctions of the problem $\Delta\Delta u-\lambda u=0$ in S, $u=\partial u/\partial n=0$ on S', where S is a bounded region in E_2 with the smooth boundary S'; let

$$E(\lambda)=\sum_{\lambda_n\leq\lambda}\phi_n^2(P),\quad I(\lambda)=\sum_{\lambda_n\leq\lambda}\phi_n(P)\phi_n(Q)\quad(P\neq Q).$$

Sharpening a result of Pleijel's [Ark. Mat. Astr. Fys. 27A, no. 13 (1940); MR 2, 291], the authors prove the estimates $E(\lambda)=(4\pi)^{-1}\lambda^{1/2}+O(\lambda^{1/4})$, $I(\lambda)=O(\lambda^{1/4})$ for $\lambda\to\infty$. In the derivation of this result the following Tauberian theorem, which is closely related to one established by Avakumović in connection with a similar problem [Math. Z. 53, 53–58 (1950); MR 12, 254] is proved: If $s(u)$ is of bounded variation on every bounded interval,

$$\int_0^\infty(u+x)^{-1}ds(u)=O(\exp\{-cx^{1/4}\})$$

for $x\to\infty$ $(c>0)$ and $s(v)-s(u)>-mu^{1/4}$ for $u\leq v\leq u+u^{3/4}$, then $s(u)=O(u^{1/4})$ for $u\to\infty$. *M. Golomb.*

Bojanić, Ranko. Propriétés asymptotiques des solutions des équations différentielles linéaires. Srpska Akad. Nauka. Zb. Rad. 35, Mat. Inst. 3, 213–254 (1953). (Serbo-Croatian. French summary)

Suppose $\lambda_1\leq\lambda_2\leq\cdots$ are the eigenvalues, $\phi_1(P)$, $\phi_2(P)$, $\cdots$ the corresponding orthonormal eigenfunctions of the problem $L(u)\equiv\Delta u-A(P)u+\lambda u=0$ in S, $u=0$ on S', where S is a bounded region in E_3, S' its smooth boundary, $A(P)\in C_1$, $A(P)>0$ for $P\in S$; let $E(\lambda)$, $I(\lambda)$ be as defined in the preceding review. The author proves the estimates

$$E(\lambda)=\lambda^{3/2}/6\pi^2+O(\lambda),\quad I(\lambda)=O(\lambda)$$

for $\lambda \to \infty$. These were obtained by Avakumović and earlier in weaker form by Pleijel [for references see preceding review] in the special case $A(P)=0$. For the general $A(P)$ the elementary solution of $L(u)=0$ is not known explicitly, but is shown to be the solution of a Fredholm equation, for which an asymptotic representation by a Stieltjes transform is derived. To this integral the following Tauberian theorem is applied: If $s(u)$ is of bounded variation on every bounded interval, $\int_0^\infty (u+x)^{-1} ds(u) = O(\exp\{-cx^{1/2}\})$ for $x \to \infty$ $(c>0)$ and $s(v)-s(u) > -m$ for $u \leqq v \leqq u+u^{1/2}$ then $s(u)=O(1)$ for $u \to \infty$.
M. Golomb (Lafayette, Ind.).

Hosemann, R., und Bagchi, S. N. Die Berechnung der Lorentz-invarianten Struktur der Greenschen Grundlösung der sog. allgemeinen Wellengleichung mittels Faltungs-operationen im Minkowski-Raum und seinem Fourier-Raum. Z. Physik 139, 1–29 (1954).

Les auteurs forment la solution élémentaire de l'opérateur des ondes $\Delta u - 4\pi^2\alpha^2 u$ invariante par le groupe de Lorentz, en utilisant des multipoles [cf. Schwartz, Théorie des distributions, t. I, Hermann, Paris, 1950; MR 12, 31] et la transformation de Fourier par rapport au temps, l'espace, et les quatre variables d'espace-temps avec le produit scalaire Lorentzien.
Y. Fourès-Bruhat (Marseille).

***Hörmander, Lars. Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 105–115 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

L'auteur considère sur une variété riemannienne à métrique hyperbolique normale continuement différentiable l'équation:

$$\Delta_2 u + (b, \operatorname{grad} u) + cu = \varphi$$

(b, c, ϕ réels et continus), dans un domain D à fermeture $\bar{D}$ compacte. La possibilité de distinguer en chaque point de D un cône caractéristique direct qui varie continuement avec le point est assurée par l'existence d'une fonction $X(u)$ dans $\bar{D}$, continuement différentiable, dont le gradient est partout orienté dans le temps.

L'auteur applique la formule de Green (la restriction $b=c=0$ est levée à la fin du papier):

$$\int_D 2(f, \operatorname{grad} u)\Delta_2 u\, du = \int_S T^{ik} f_i dS_k - \int_D T^{ik} f_{i,k} dx,$$

$$T_{ik} = 2\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^k} - g_{ik}\Delta_1 u$$

et montre l'existence de champ de vecteurs f orientés dans le temp tels que: a) $T^{ik} f_{i,k}$ est définie positive, b) l'intégrale de surface est non positive si les donnés de Cauchy sur une portion S^+ de S orientée dans l'espace ou caractéristique et les valeurs aux limites sur la portion S_t (satisfaisant à une condition de régularité) sont nulles. D'où un théorème d'unicité. Majoration des solutions: généralisation d'inégalités de Friedrichs-Lewy et Schauder-Krzyżański par estimation de la semi-norme

$$(u, u)_1 = \int_D T^{ik} f_{i,k} dx.$$

Une partie des résultats s'étend au cas ultrahyperbolique.
Y. Fourès-Bruhat (Marseille).

Protter, M. H. The characteristic initial value problem for the wave equation and Riemann's method. Amer. Math. Monthly 61, 702–705 (1954).

The variation of Riemann's method first employed by Martin [Bull. Amer. Math. Soc. 57, 238–249 (1951); MR 13, 244] is used by the author for the wave equation in three variables (*) $u_{xx}+u_{yy}=u_{zz}$ and the characteristic cone (**) $x^2+y^2=(z-1)^2$. The problem is to find a solution of (*) interior to (**), satisfying the condition

$$u(x, y, 1-(x^2+y^2)^{1/2}) = \psi(x, y), \quad 0 \leqq x^2+y^2 \leqq 1.$$

The given function $\psi(x, y)$ is assumed to have continuous second derivatives. The author gets the equation

$$(***) \qquad Lu \equiv u_{\alpha\beta} - \frac{1}{2(\alpha-\beta)}(u_\alpha - u_\beta) - \frac{u_{\Phi\Phi}}{(\alpha-\beta)^2} = 0$$

by introducing (α, β, Φ) as new variables according to the relations

$$x = \tfrac{1}{2}(\alpha-\beta)\cos\Phi, \quad y = \tfrac{1}{2}(\alpha-\beta)\sin\Phi, \quad z = \tfrac{1}{2}(\alpha+\beta).$$

The associate equation

$$Mv \equiv v_{\alpha\beta} + \frac{1}{2(\alpha-\beta)}(v_\alpha - v_\beta) = 0$$

assumes the role of the adjoint equation usually considered in such problems. Now the problem can be solved by an explicit formula and the method seems much simpler than that given by R. Courant and D. Hilbert [Methoden der mathematischen Physik, vol. II, Springer, Berlin, 1937] which depends on a mean-value theorem of Asgeirsson and the solution of an Abel integral equation.
M. Pinl.

Ludford, G. S. S. Riemann's method of integration: its extensions with an application. Collect. Math. 6, 293–323 (1953).

§1 of this paper contains the canonical form of the second-order hyperbolic differential equation in two independent variables. The adjoint operator M of the operator $L(w) \equiv w_{rs} + aw_r + bw_s + cw$ is defined by

$$M(v) \equiv v_{rs} - (av)_r - (bv)_s + cv$$

(§2). In the Cauchy problem of the first kind the values of w and $\partial w/\partial n$ are prescribed on an arc C which is nowhere tangential to a characteristic. The solution of this problem is given by the formula of Riemann (§3). Its discussion is given in §4. Next, §5 contains examples of the Riemann function: (1) $a=b=(r+s)^{-1}$, $\lambda=$const., $c=0$; (2) $a=b=m=$const., $c=0$. In the Cauchy problem of the second kind the values of w only are prescribed on two segments of characteristic lines of different families. The solution is given again by Riemann's formula (after some modifications (§6)). §7 contains an extension of Riemann's method to the case where the curve C is tangential to a characteristic at some point. The application of Riemann's method is successful in the case of one-dimensional gas dynamics (§§8, 9). In §§10, 11, 12 the special case of motion in a closed tube is discussed and an estimate of the time of first occurrence of breakdown has been made. Finally, some remarks and supplements are given (§§13, 14, 15) regarding especially two general theorems used by H. Rademacher in the verification of Riemann's fundamental formula.
M. Pinl (Cologne).

Vasilache, Sergiu. Une nouvelle équation des télégraphistes. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 295–320 (1952). (Romanian. Russian and French summaries)

Vasilache, Sergiu. Sur une nouvelle équation des télégraphistes. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 4, 373–393 (1953). (Romanian. Russian and French summaries)

In the first one of these two papers the author establishes the system of partial differential equations of the first order satisfied by the voltage and current in a transmission line in which the inductance and capacitance are not constant (as usually assumed) but periodic functions of time. These functions are assumed to be of the form

$$L=L_0(1+m\cos\omega t),\quad C=C_0(1+n\cos\omega t).$$

He obtains a partial differential equation of the second order by eliminating the current, determines and investigates the characteristics of this equation (which are no longer straight lines), transforms the partial differential equation to characteristic coordinates, and writes down the solution by Riemann's method. Riemann's function is determined by an integral equation.

Four problems are treated explicitly. (i) Semi-infinite transmission line with given voltage at the beginning. (ii) Finite line, given voltage at one end, closed over a given impedance at the other end. (iii) Finite line, given voltage at one end, short circuited at the other end. (iv) Finite line, given voltage at one end, open at the other end.

In the second paper another method of attack is developed. Instead of integrating the partial differential equation by Riemann's method, the author converts this equation into an integral equation, and discusses the solution of the integral equation. Problem (i) above is treated in detail. *A. Erdélyi* (Pasadena, Calif.).

Fleishman, Bernard A. On the periodic solutions to an initial-value problem for a Duffing-type nonlinear wave equation. Applied Physics Laboratory, the Johns Hopkins University, Silver Spring, Md., Bumblebee Rep. No. 209, v+24 pp. (1954).

The author proves the existence of solutions periodic in time of the nonlinear and inhomogeneous hyperbolic equation $u_{xx}=u_{tt}+ku_t+\alpha u+\epsilon u^3+B$; the functions u and B are assumed periodic in space as well as time. The parameters k, α, ϵ, and the space and time periods have to satisfy certain conditions which, roughly, amount to requiring the damping coefficient k to be large enough. The effect of these conditions is to insure that the mapping of the Cauchy data (u, u_t) from $t=0$ to $t=p$ by solutions of the above equation contracts in the (C_2, C_1) norm. The estimates needed to prove the contraction are Haar-type estimates. *P. D. Lax* (New York, N. Y.).

Douglis, Avron. Observations on normal forms of linear hyperbolic equations of second order. Comm. Pure Appl. Math. 7, 675–695 (1954).

Dans une note précédente [mêmes Comm. 7, 271–295 (1954); MR 16, 44], l'auteur indique comment l'introduction de coordonnées convenables permet de ramener à une forme normale l'équation linéaire hyperbolique du second ordre à coefficients variables, dont les coefficients des termes du second ordre sont deux fois continument différentiables. Dans le présent travail, il ajoute quelques compléments en vue de l'étude de la validité du principe d'Huygens. D'une part, il étudie les relations entre les formes normales des équations obtenues en multipliant préalablement une équation donnée par un facteur adéquat. D'un autre côté, il étudie les transformations qui font passer d'un système de coordonnées normales en un autre, au même point, généralisations de la transformation de H. A. Lorentz pour l'équation des ondes à coefficients constants. *H. G. Garnir* (Liège).

Elianu, I. P. Le problème de Cauchy pour les équations aux dérivées partielles linéaires et polyhyperboliques normales. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 367–474 (1952). (Romanian. Russian and French summaries)

Let F be a linear partial differential operator of the second order in n variables, with analytic coefficients, and let F^p be the pth power of this operator. The author studies the partial differential equation $F^pu=0$. With this equation the author associates the Riemannian metric usually associated with $Fu=0$. Solutions of $F^pu=0$ of the form $u=UG^q$, where where U is regular, are possible only if $G=0$ is a characteristic hyper-surface. The bicharacteristics are null geodesics of the Riemannian space, characteristic conoids $\Gamma=0$ are introduced, and the elementary solution of $F^pu=0$ is constructed following Hadamard. In the case of an odd number ($n=2m+1$) of variables the elementary solution is of the form $U\Gamma^{p-m-1/2}$. In the case of an even number of variables, the elementary solution contains logarithmic terms, but the form of these terms depends on whether $n>2p$ or $n\leqq 2p$. The analytic parts (such as U in the case of odd n) are constructed as infinite series whose convergence is proved by majorization.

In the second part the author discusses the regular Cauchy problem for the equation $F^pu=\phi$, and its solutions by Hadamard's method, utilizing the finite (or logarithmic) part of a divergent integral. An examination of the solution shows that Huygens' minor premise (in the sense of Hadamard) holds if n is even, $n>2p$, and the elementary solution contains no logarithmic terms.

In the third and last part the cases $F=\Box$ and $F=\Box-k$ are discussed, where $\Box$ is the d'Alambertian,

$$\left(\frac{\partial}{\partial x^1}\right)^2-\sum_{r=2}^{n}\left(\frac{\partial}{\partial x^r}\right)^2.$$

A. Erdélyi (Pasadena, Calif.).

Morawetz, Cathleen S. A uniqueness theorem for Frankl's problem. Comm. Pure Appl. Math. 7, 697–703 (1954).

L'équation considérée est $K(y)u_{xx}+u_{yy}=0$; K est une fonction de y à dérivée positive et $K(0)=0$. Le domaine où est définie la solution est limité par un arc simple C_0 dans $y>0$ dont les extrémités A et B sont sur l'axe des x, $x(A)\leqq 0<x(B)$, deux arcs de caractéristiques issus de 0 et deux arcs C_1 et C_2 issus respectivement de A et B, ayant des orientations d'espace, si la direction des y négatifs est prise comme axe des temps, dont les extrémités sont situées sur les deux caractéristiques issues de 0. Les valeurs de u sont données sur les arcs C_0, C_1, C_2. L'unicité de la solution de l'équation satisfaisant à ces conditions est démontrée, moyennant des hypothèses assez larges sur la forme de C_0, les dérivées de u pouvant même admettre certaines singularités au voisinage des points 0, A, B. La méthode consiste à former une identité satisfaite pour toute solution de l'équation, contenant deux fonctions arbitraires; un choix judicieux de ces fonctions permet de conclure à l'unicité de la solution d'un tel problème. *P. Germain* (Paris).

Germain, P. Remarks on the theory of partial differential equations of mixed type and applications to the study of transonic flow. Comm. Pure Appl. Math. 7, 117–143 (1954).

This paper is divided into two parts. In the first part various problems that arise in the study of transonic flow are discussed. A clear exposition is given of the problems of transonic flow in a nozzle and flow past a two-dimensional profile. It is shown how these problems lead, by the hodograph method (or the method of Legendre's potential), to rather complicated boundary-value problems for equations of mixed type.

Part II of the paper discusses some of the author's mathematical work in the theory of equations of the form (*) $K(z)u_{zz}+u_{ss}=0$, where $K(z)$ is a monotone increasing function with $K(0)=0$. The problem of finding a fundamental solution of equation (*) is solved for the case where the singularity may occur either in the elliptic or the hyperbolic portion of the plane. This generalizes the work of many authors who found fundamental solutions when the singularity is in the elliptic portion of the plane. It also extends the results of the author and R. Bader [O.N.E.R.A. Publ. no. 54 (1952); MR 14, 654] who previously considered the same problem for the case $K(z)=z$. A detailed analysis is given of the behavior of the elementary solution when the singularity is in the hyperbolic portion of the plane. Here the singularity propagates along characteristics and the effect of the parabolic line $z=0$ must be taken into account.

In the final section of the second part the author discusses boundary-value problems with the aid of the fundamental (or elementary) solution just obtained. First it is shown how certain singular problems can be solved when the domain is purely hyperbolic or purely elliptic with part of the boundary along the parabolic line. The determination of the Green's function for the Tricomi problem is reduced to the "conjugate Tricomi problem". The conjugate Tricomi problem is analogous to the ordinary Tricomi problem in that the data are given along the elliptic arc and a bounding characteristic of the domain. With the aid of this result it is shown that the existence of the solution of the conjugate problem implies the uniqueness of the solution of the Tricomi problem.

As the author states the boundary problems that arise in transonic flows are more complicated than the boundary problems that have been studied mathematically thus far. The present work, however, makes a significant advance toward the study of such complicated problems.

M. H. Protter (Berkeley, Calif.).

Èĭdel'man, S. D. On a connection between the fundamental matrices of solutions of parabolic and elliptic systems. Mat. Sb. N.S. 35(77), 57–72 (1954). (Russian)

Considérons un système différentiel parabolique (au sens de Petrowsky):

$$(*)\qquad \frac{\partial}{\partial t}+\Phi^{(2b)},$$

où $\Phi^{(2b)}$ est un opérateur différentiel elliptique d'ordre $2b$ en variables d'espaces, supposé à coefficients constants. Il y a évidemment une relation entre la matrice fondamentale de (*) et la solution élémentaire de $\Phi^{(2b)}$. L'auteur explicite cette relation (formule (31)); il utilise pour cela les majorations précises obtenues pour la matrice fondamentale de (*) dans un travail antérieur [Mat. Sb. N.S. 33(75) 359–382 (1953); il utilise, sans les rappeler, les notations de cet article; MR 15, 712]. A titre d'example, l'auteur donne la solution élémentaire des équations de l'élasticité.

J. L. Lions (Nancy).

Pini, Bruno. Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico. Rend. Sem. Mat. Univ. Padova 23, 422–434 (1954).

Let γ_1, γ_2 denote the two curves $x=\chi_1(y)$, $x=\chi_2(y)$, $a\leqq y\leqq b$, with $\chi_1(y)<\chi_2(y)$ for $a<y\leqq b$. $C(\alpha)$ will be used to denote the line segment $\chi_1(\alpha)\leqq x\leqq\chi_2(\alpha)$, $y=\alpha$; and D denotes the closed region bounded by $\gamma_1+C(a)+\gamma_2+C(b)$. The curve $S=\gamma_1+C(a)+\gamma_2$ is called the parabolic contour associated with region D. If the functions $\chi_i(y)$ satisfy Hölder conditions of order $>\frac{1}{2}$, then for the region D there exists a unique solution of the first boundary-value problem for the parabolic equation. That is a function $u=u(x,y)$ such that

$$(1)\qquad L(u)\equiv u_{xx}-u_y=0 \quad (\text{in } D-S),\qquad u=f \quad (\text{on } S),$$

where f is a preassigned continuous function. In what follows $D(y_1,y_2)$ will indicate the closed subset of D which lies in the strip bounded above by $C(y_2)$ and below by $C(y_1)$. Let $\chi_2(y)$ belong to class C^1, but assume only that $\chi_1(y)$ is continuous in order to introduce the idea of an irregular point. The point $(\chi_1(y_0),y_0)$ will be called an irregular boundary point if there exists a y_1 $(a\leqq y_1<y_0)$ such that for every y_2 $(y_1<y_2<y_0)$ the first boundary-value problem for $D(y_1,y_2)$ is always solvable but for $D(y_1,y_0)$ it is not always solvable. A boundary point is called regular if it is not irregular. Similar definitions hold for points on γ_2.

Assuming only continuity of the functions $\chi_i(y)$, the author proves the existence of a so-called generalized solution $U=U(x,y)$ of problem (1). In $D-S$ the function U satisfies $L(U)=0$, and at each regular point P of the boundary S the function U assumes the value $f(P)$. The generalized solution is of course the ordinary solution in case all points on the boundary S are regular. Necessary and sufficient conditions that the first boundary-value problem for D has a solution are given in terms of so-called barrier functions. The author gives a set of conditions which, if satisfied, insure regularity of a boundary point; conditions insuring irregularity are also given.

F. G. Dressel.

Meĭman, N. N. On the equation of heat conduction. Dokl. Akad. Nauk SSSR (N.S.) 99, 209–212 (1954). (Russian)

The author applies the methods developed by him in an earlier paper on difference equations [same Dokl. (N.S.) 97, 593–596 (1954); MR 16, 254] to an initial-value problem for the quasi-linear parabolic equation

$$(*)\qquad L(u)=\frac{\partial u}{\partial t}-D(x,t,u)\frac{\partial^2 u}{\partial x^2}-A(x,t,u)\frac{\partial u}{\partial x}-B(x,t,u)u-Q(x,t,u)=0.$$

It is assumed that in the strip S: $-\infty<x<\infty$, $0\leqq t\leqq T$, the coefficients satisfy the relations, $0\leqq D\leqq d$, $|A|<a$, $|B|<b$, $|Q|<q$, for appropriate constants d, a, b, q, and all values of u. A solution $u(x,t)$ of (*) in S, which for $t=0$ assumes prescribed initial values $\varphi(x)$, is called regular if it is bounded and uniformly continuous in S together with all derivatives entering in (*).

For the case that the coefficients of (*) are independent of the unknown function u, the author proves the theorems: 1) if the coefficients are uniformly continuous in S, and if $\varphi(x)$, together with its derivatives up to the second order, is bounded and uniformly continuous, then there is at most one regular solution $u(x, t)$ in S. If such a solution exists, then the solutions of an appropriately defined difference scheme tend uniformly to $u(x, t)$ as the intervals of the net tend to zero; 2) if, in S, the coefficients and initial function have bounded derivatives in x of the first three orders which satisfy a uniform Lipschitz condition, then there is one and only one regular solution $u(x, t)$ in S. If $T=\infty$, the solution exists for the entire half plane $t\geq 0$.

For coefficients which may depend on u, the author proves: 3) if the hypotheses of 2) are satisfied in both x and u, there is a unique regular solution $u(x, t)$ in a substrip $0\leq t\leq T_{NR}\leq T$ of S. The author makes the following remarks: a) if the restrictions on the coefficients are weakened the method leads to the construction of a generalized solution; b) the method applies without change to equations with more than one space variable; c) the method can be adapted to the solution of the initial and boundary value problem for the equation (*). The demonstrations are based on a discussion of the stability of appropriate difference schemes, in accordance with a definition proposed by the author in the paper cited above. The results are closely related with those given by Oleĭnik and Ventcel' [ibid. 97, 605–608 (1954); MR 16, 259] who use an iteration procedure.

R. Finn (Los Angeles, Calif.).

Lovass-Nagy, Viktor, Pál, Sándor, und Pásztor, János. Untersuchung einiger die Erwärmung eines durch Induktion erhitzten kreiszylinderförmigen Körpers betreffenden Fragen. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 499–511 (1954). (Hungarian. Russian and German summaries)

The authors consider heat conduction in a circular cylinder when the temperature $u(r, t)$ depends only on the distance from the axis of the cylinder and on time, $u(r, 0)=0$, and $\partial u(r, t)/\partial r$ is given on the surface of the cylinder. The solution is obtained as the sum of an elementary function and an infinite series of Bessel functions. The results of a numerical evaluation of this solution are shown in diagrams, and some technological applications are indicated.

A. Erdélyi (Pasadena, Calif.).

Náter, I. Solution of the temperature relations in a beam with discontinuous boundary conditions. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 70–78 (1954). (Slovak. Russian summary)

The author determines the stationary temperature distribution in a circular cylinder of finite length when the lower half of the surface of the cylinder is kept at a constant temperature θ_1, while the upper half of the surface is at temperature θ_2. [The reviewer is unable to verify the correctness of the solution.]

A. Erdélyi.

Minasyan, R. S. On a problem of heat conduction. Akad. Nauk Armyan. SSR. Dokl. 12, 65–71 (1950). (Russian. Armenian summary)

The author considers the problem of the steady-state distribution of temperature in a plane homogeneous medium. The region concerned is an L-shaped region R which is bounded by the straight lines:

$$y=0,\quad 0\leq x\leq b;\quad x=0,\quad 0\leq y\leq b;\quad y=b,\quad 0\leq x\leq d;$$
$$x=b,\quad 0\leq y\leq d;\quad y=d,\quad d\leq x\leq b;\quad x=d,\quad d\leq y\leq b.$$

The temperature function satisfies the equation $U_{xx}+U_{yy}=0$ in R, and boundary conditions

$$\begin{gathered} U_x(d, y)+hU(d, y)=0 \quad (y\geq d),\\ U_y(x, d)+hU(x, d)=0 \quad (x\geq d),\\ -U_x(0, y)+hU(0, y)=0,\quad -U_y(x, 0)+hU(x, 0)=0,\\ U(b, y)=P(y),\quad U(x, b)=P(x). \end{gathered}$$

The method of solution consists of breaking R into three sub-regions, finding a generalized Fourier series solution in each sub-region subject to appropriate boundary conditions and matching these solutions along the lines of subdivision.

C. G. Maple (Ames, Iowa).

Minasyan, R. S. On the plane steady distribution of temperature in nonhomogeneous prismatic bodies. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki 5, no. 5, 1–24 (1952). (Russian. Armenian summary)

A square of side $2d$ is imbedded in a square of side $2b$ $(b>d)$ such that the corresponding sides are parallel. A coordinate system with its origin at the common center of these squares and axes parallel to the sides is chosen. The area between the two squares and the area totally within the smaller square are designated by R_1 and R_2, respectively. If λ_1 and λ_2 are the coefficients of heat conduction associated with the material occupying R_1 and R_2, then the corresponding temperatures $U_i(x, y)$ in R_i satisfy the equations

$$\frac{\partial^2 U_i(x, y)}{\partial x^2}+\frac{\partial^2 U_i(x, y)}{\partial y^2}=-\frac{1}{\lambda_i}Q_i(x, y) \quad (i=1, 2) \tag{1}$$

with the boundary conditions

$$U_1(x, \pm b)=T(x),\quad U_1(\pm b, y)=T(y), \tag{2}$$

where $Q_1(x, y)$ represent the intensity of the heat source in the ith medium. The conditions along the common boundaries are

$$\begin{gathered} U_1(x, \pm d)=U_2(x, \pm d),\quad \lambda_1\frac{\partial U_1}{\partial y}\bigg|_{y=\pm d}=\lambda_2\frac{\partial U_2}{\partial y}\bigg|_{y=\pm d},\\ U_1(\pm d, y)=U_2(\pm d, y),\quad \lambda_1\frac{\partial U_1}{\partial x}\bigg|_{x=\pm d}=\lambda_2\frac{\partial U_2}{\partial x}\bigg|_{x=\pm d}. \end{gathered} \tag{3}$$

Because of the geometrical symmetry and the symmetry in the boundary conditions (2), it is sufficient to consider the problem of the temperature distribution in that quarter of the configuration which is within the first quadrant only with the additional conditions

$$\frac{\partial U_i}{\partial x}\bigg|_{x=0}=\frac{\partial U_i}{\partial y}\bigg|_{y=0}=0. \tag{4}$$

The function $U_1(x, y)$ is taken of the form

$$U_1(x, y)=\begin{cases} V_1(x, y) & \text{for}\quad x\geq d,\quad y\leq d,\\ V_2(x, y) & \text{for}\quad x\geq d,\quad y\geq d,\\ V_3(x, y) & \text{for}\quad x\leq d,\quad y\geq d. \end{cases} \tag{5}$$

Because of the symmetry $V_3(x, y)=V_1(y, x)$. The functions $V_i(x, y)$ are subject to the appropriate boundary and matching conditions. The method of solution of this problem is one in which a generalized Fourier series is found for each of the functions $V_i(x, y)$ $(i=1, 2, 3)$ and $U_2(x, y)$. These are subjected to the various boundary conditions to determine the characteristic numbers and the coefficients. An example is given in which $Q_1(x, y)=0$, $Q_2(x, y)=Q_2$, $T(x)=T(y)=T$, $\lambda_2=5\lambda_1$, $d=\frac{2}{3}b$.

C. G. Maple (Ames, Iowa).

Integral Equations

Schaefer, Helmut. Zur Theorie nichtlinearer Integralgleichungen. Math. Nachr. 11, 193–211 (1954).

This paper, the author's doctoral thesis, is in method and results close to an earlier paper by the reviewer [Math. Z. 39, 45–75 (1934)]. A typical result is the following: Every continuous solution of the system of integral equations $x_i(s)=\int_B K_i(s,t)f_i(t,x_1(t),\cdots,x_r(t))\,dt$ $(i=1,\cdots,n)$, where the f_i satisfy the Lipschitz condition

$$\int_B |f_i(t,x_1,\cdots,x_n)-f_i(t,y_1,\cdots,y_n)|^2dt \leqq L^2\max\int_B|x_j-y_j|^2dt$$

with a Lipschitz constant L that may depend on

$$Q=\max\left(\int|x_j|^2dl,\ \int|y_j|^2dt\right),$$

can be determined by solving a certain finite system of transcendental equations together with a modified system of integral equations whose solution is unique and can be found by the method of successive approximation. An application is made to Duffing's problem. *M. Golomb.*

Ahiezer, N. I. On some dual integral equations. Dokl. Akad. Nauk SSSR (N.S.) 98, 333–336 (1954). (Russian)

Solution of the dual integral equation

$$\int_k^\infty S(y)J_0(xy)y\,dy=f(x)\quad(0<x<1),$$

$$\int_0^\infty S(y)J_0(xy)(y^2-k^2)^p y\,dy=0\quad(x>1),$$

and also of

$$\int_k^\infty C(y)J_\nu(xy)\,dy=f(x)\quad(0<x<1)$$

$$\int_0^\infty C(y)J_{-\nu}(xy)\,dy=0\qquad(x>1),$$

for $k\geqq 0$, $-1<p<1$, $-1<\nu<1$. *A. Erdélyi.*

Kumar, Ram. A theorem on integral equation. Gaṇita 4, 123–128 (1953).

Under certain conditions on the parameters, the author shows that $f(x)$ satisfies the integral equation

$$f(x)=x^\lambda\int_0^\infty e^{ax-by}J_\lambda{}^\mu(x^\mu y)f(y)dy,$$

where

$$J_\lambda{}^\mu(z)=\sum_{r=0}^\infty\frac{(-z)^r}{r!\Gamma(1+\lambda+\mu r)},$$

provided that $\phi(p)$, the Laplace transform of $f(x)$, satisfies

$$\phi(p)=(p-a)^{-\lambda-1}\phi(b+(p-a)^{-\mu}).$$

He gives examples of this theorem. *A. Erdélyi.*

Busbridge, I. W., and Stibbs, D. W. N. On the intensities of interlocked multiplet lines in the Milne-Eddington model. Monthly Not. Roy. Astr. Soc. 114, 2–16 (1954).

In this paper the system of equations

$$\mu\frac{dI_r(\tau,\mu)}{d\tau}=(1+\eta_r)I_r(\tau,\mu)-\tfrac{1}{2}(1-\epsilon)\alpha_r\sum_{p=1}^k\eta_p\int_{-1}^{+1}I_r(\tau,\mu')d\mu' -(1+\epsilon\eta_r)(a+b\tau)\quad(r=1,2,\cdots,k),$$

where ϵ, η_r, a and b are constants and $\alpha_r=\eta_r/\sum_{p=1}^k\eta_p$, is solved by the method derived from the principles of invariance [cf. S. Chandrasekhar, Radiative transfer, Oxford, 1950; MR 13, 136]. It is shown in particular that the angular distribution of the emergent radiation can be expressed in terms of the H-function defined by

$$H(\mu)=1+\tfrac{1}{2}\mu H(\mu)\sum_{p=1}^k\alpha_p(1-\lambda_p)\int_0^1\frac{H(\mu')}{\mu+\mu'}d\mu',$$

where $\lambda_p=(1+\epsilon\eta_p)/(1+\eta_p)$, and its moments. The case $r=2$ is considered in some detail and the required H-function is tabulated for $\eta_1=10$, $\eta_2=5$; $\eta_1=4$, $\eta_2=2$ and $\eta_1=1$, $\eta_2=\tfrac{1}{2}$. Various astrophysical applications of these solutions are considered. *S. Chandrasekhar* (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

Hirasawa, Yoshikazu. On Newton's method in convex linear topological spaces. Comment. Math. Univ. St. Paul. 3, 15–27 (1954).

Let Y and Z be complete locally convex linear topological spaces over the complexes, and let T be an "analytic" function defined on an open set of Y to Z. It is desired to solve the equation $T(y)=0$ by means of Newton's iterative process $y_{n+1}=y_n-\delta T^{-1}(y_n;T(y_n))$, $n=0,1,2,\cdots$. Here the expression on the right denotes the inverse of a suitably defined differential of T at the point y_n with increment $T(y_n)$. The author gives conditions which are sufficient to assure the convergence of the sequence $\{y_n\}$ to a solution, but they cannot be stated here. The uniqueness is obtained under stronger conditions. This work is close in spirit and content to a paper of M. L. Stein [Proc. Amer. Math. Soc. 3, 858–863 (1952); MR 14, 1094], which it generalizes. In order to establish his result the author briefly extends the basic theory of analytic functions to complete complex convex linear topological spaces to obtain the Taylor expansion. [For the Banach space case see E. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948, Chap. IV; MR 9, 594.] *R. G. Bartle.*

Marinescu, G. Opérations relativement complètement continues. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, 107–194 (misprinted 107–188) (1951). (Romanian. Russian and French summaries)

The author extends to quasi-normed linear spaces the Riesz theory of completely continuous transformations as well as various ergodic theorems. The notion of complete continuity is replaced by that of relative complete continuity (r.c.c.). A quasi-norm has the usual properties of a norm except that the triangle inequality is replaced by the condition that there exists a number $\mu\geqq 1$ such that for all vectors x, y one has $|x+y|\leqq\mu(|x|+|y|)$. An r.c.c. transformation is defined as follows: Let L be a quasi-normed linear space and let $\mathbf{L}$ be a linear subspace of L, not necessarily closed. Let $\mathbf{L}$ also have defined on it a quasi-norm, giving rise thus to two norms for each x in $\mathbf{L}$, indicated by $|x|_L$ and $|x|_{\mathbf{L}}$. Then T is relatively completely continuous if: $T(L)\subset L$, $T(\mathbf{L})\subset\mathbf{L}$; T is continuous from L to L, from $\mathbf{L}$ to L, and from $\mathbf{L}$ to $\mathbf{L}$. Transforms every bounded set in $\mathbf{L}$ in a compact set in L. The Riesz theory is then established for r.c.c. transformations which satisfy additional conditions. The list of these conditions cannot be given here since it is very long. There are eight ergodic theorems given in chapter III. A

typical one is: Let T be linear from L to L and suppose that for all z, $n^{-1}|T^n z|\to 0$. Consider the linear manifold of x such that a) the sequence $\{x_n\}$, $nx_n=(T+T^2+\cdots+T^n)x$, has a subsequence convergent to $\bar{x}$, b) $\bar{x}-x \in (I-T)L$. Then $\{x_n\}$ converges to $\bar{x}$ and the transformation $T_1x=\bar{x}$ satisfies $TT_1=T_1T=T_1^2=T_1$. The remaining chapters give applications and relate the present work to previous papers of the author. *E. R. Lorch* (New York, N. Y.).

Povolockiĭ, A. I. On the existence of disconnected spectra for nonlinear completely continuous operators. Dokl. Akad. Nauk SSSR (N.S.) **99**, 345–348 (1954). (Russian)

Notations: E a Banach space; A a completely continuous operator in E; Λ = spectrum of

$$A=\{\lambda\,|\,Ax=\lambda x,\ x\neq\theta\};\quad \Pi_{r,R}=\{x\,|\,r<|x|<R,\ x\in E\}.$$
$$\Lambda_{r,R}=\{\lambda\,|\,Ax=\lambda x,\ x\in\Pi_{r,R}\};$$

condition

$$(*)\qquad |Ax-Bx|\leqq K|x|,\quad x\in\Pi_{r,R},$$

B a linear completely continuous operator;

$$\mathfrak{N}_{(\alpha,\beta)}=\{x\,|\,x\in\Pi_{r,R},\ Ax=\lambda x,\ \alpha<\lambda<\beta\};$$

$\mathfrak{N}_{(\alpha,\beta)}$ is a continuous branch in $\Pi_{r,R}$ if $\mathfrak{N}_{(\alpha,\beta)}\cap L\neq 0$ for all bounded regions L such that $\{x\,|\,|x|\leqq r\}\subset L\subset\{x\,|\,|x|<R\}$;

$$E_\lambda=\{x\,|\,Bx=\lambda x\};$$
$$\mathcal{E}_{\lambda,m}=\{x\,|\inf\{|x-e|/e\in E_\lambda\}<m|x|\};$$

B_θ = Fréchet derivative of A at θ; $\lim_{|x|\to 0}|Ax-B_\infty x|/|x|=0$, if B_θ and B_∞ exist. Results (no proofs offered): 1. Let A satisfy condition (*) in $\Pi_{0,\infty}$; and

$$(**)\quad K<\min\{|((\lambda_0-\epsilon)I-B)^{-1}|^{-1};\ |((\lambda_0+\epsilon)-B)^{-1}|^{-1}\}.$$

If in $(\lambda_0-\epsilon, \lambda_0+\epsilon)$, λ_0 is the only proper value of B and if λ_0 is simple and if B_0 and B_∞ have only one proper value μ, resp. ν, in the same interval, then $\mathfrak{N}_{(\lambda_0-\epsilon,\lambda_0+\epsilon)}$ is a continuous branch in $\Pi_{0,\infty}$ and $\Lambda\cap(\lambda_0-\epsilon,\lambda_0+\epsilon)\subset(\mu,\nu)$. 2. When applied to the operator $A\varphi(s)=\int_G K(s,t)f(\varphi(t))dt$, where $\int_G\int_G K^2(s,t)\,dsdt=1$, $K(s,t)=K(t,s)$, positive definite, G a compact set of measure 1 in n-space, f continuous on $(-\infty,\infty)$, we get: Let $B\varphi(s)=\int_G K(s,t)\varphi(t)\,dt$ have proper values $\lambda_1>\lambda_2>\cdots>\lambda_n$ which are simple and such that $[\lambda_1,\lambda_n]$ contains no other proper values of B. Furthermore, let (a) $|u-f(a)|<2\alpha$, (b) $f'(0)=1$, $\lim_{x\to\infty}f(x)/x=1-\alpha$, where $\alpha<\min_{i,j}\frac{1}{2}|\lambda_i-\mu_{i-j}|;\ 1-(\mu_i/\lambda_i)$, where

$$\mu_0>\lambda_1>\mu_1>\lambda_2>\cdots>\lambda_n>\mu_n.$$

Then $(\lambda_i,(1-\alpha)\lambda_i)$ contain different connected components of Λ. 3. Let A satisfy (2) in $\Pi_{0,R}$, λ_0 the only proper value of B in $(\lambda_0-\epsilon,\lambda_0+\epsilon)$, multiplicity α. Let (**) be fulfilled. Then $\lambda_0\pm\epsilon$ are not in the point spectrum of B_θ and the sum of the multiplicities of the points in the point spectrum of B_θ which are in $(\lambda_0-\epsilon,\lambda_0+\epsilon)$ is α. Other results in the paper have a similar character. *B. Gelbaum.*

Aronszajn, N., and Smith, K. T. Invariant subspaces of completely continuous operators. Ann. of Math. (2) **60**, 345–350 (1954).

If T is a bounded linear transformation of a Banach space $\mathfrak{B}$ into itself, the problem of finding proper invariant subspaces of T is that of finding subspaces $\mathfrak{L}$, $(0)\neq\mathfrak{L}\neq\mathfrak{B}$, for which $T(\mathfrak{L})\subset\mathfrak{L}$. The authors prove that if T is a completely continuous operator in $\mathfrak{B}$, then T has a proper invariant subspace. (The result gives new information only for quasi-nilpotent transformations—those whose spectrum consists of $\lambda=0$ only.) Fundamental to the proof is the notion of "metric projection, P, on a finite-dimensional $\mathfrak{L}$". Assuming, as one may, that $\mathfrak{B}$ is strictly convex, P_x is defined for every x as the unique point of $\mathfrak{L}$ which minimizes the distance $\|y-x\|$ with y in $\mathfrak{L}$. The paper includes Aronszajn's brief proof of the theorem for Hilbert space, which is similar to an unpublished earlier proof of von Neumann. *E. R. Lorch* (New York, N. Y.).

Schreiber, Shmuel. Tailpiece to a result by Calabi and Dvoretzky. Riveon Lematematika **8**, 30–31 (1954). (Hebrew. English summary)

Theorem. Let $\{G_m\}$ be a sequence of closed convex sets in a normed linear space, all containing the origin and of diameter not exceeding 2. Then there exists a sequence of elements $\{p_m\}$, with p_m on the boundary of G_m, such that $\|\sum_{m=1}^n p_m\|\leqq 1$ for all n. The proposition referred to in the title [Trans. Amer. Math. Soc. **70**, 177–194 (1951); MR **12**, 604] is that if the G_m's are triangles in the complex plane, then there are vertices p_m such that $|\sum_{m=1}^n p_m|\leqq\sqrt{5}$. *M. Jerison* (Lafayette, Ind.).

Feller, William. The general diffusion operator and positivity preserving semi-groups in one dimension. Ann. of Math. (2) **60**, 417–436 (1954).

The author gives an abstract definition of a diffusion process and the most general form of a diffusion operator in one dimension. Let E be (finite or infinite) closed interval on the real line and denote by M_+ the set of all non-negative measures on the Borel sets of E. A process is, by definition, a one-parameter semi-group $\{T_t\}_{t>0}$ of linear transformations on M_+ into M_+ such that the convergence $\mu_n\to\mu$ of measures as functionals on $C(E)$ implies $\mu_nT_t\to\mu T_t$. It is to be noted that the author does not assume that T_t preserves the measure. Theorem 1 states that such a process is in one-one correspondence with the semi-group $\{T_t\}_{t>0}$ of non-negativity-preserving bounded linear transformations on $C(E)$ into $C(E)$. A process is called of local character, or a diffusion process, if (i) for each $f\in C(E)$ vanishing identically in some neighbourhood of any point $x_0\in E$ one has $\lim_{t\downarrow 0}t^{-1}(T_tf-f)(x_0)=0$ and (ii) for each $t>0$ $\lim_{h\downarrow 0}(T_{t+h}f)(x)=(T_tf)(x)$ uniformly in E. The local infinitesimal generator A of the diffusion process T_t is defined by the local bounded convergence $\lim_{t\downarrow 0}t^{-1}(T_tF-F)(x)=f(x)$, where $F\in C(E)$ and f is continuous in some neighbourhood of some point $x_0\in E$. In such case the author writes $F\in D(A;x_0)$. The lemma 2 plays a central role in the present paper: Let $F\in D(A;x_0)$; if $F(x_0)=0$ and F has a local minimum at x_0, then $(AF)(x_0)\geqq 0$.

The point x_0 is called a null point if $(T_tf)(x_0)=0$ for each $f\in C(E)$ and all $t>0$. The absorbing barriers in classical diffusion are null points. The point x_0 is called an absorption point if for each $F\in D(A;x_0)$ we have $(AF)(x_0)=cF(x_0)$. The point x_0 is called a right (left) translation point if $A(f(x)g(x_0)-f(x_0)g(x))>0$, $|x-x_0|<\epsilon$, implies that in some neighbourhood of x_0 the ratio f/g strictly increases (decreases). The main result is summed up in theorem 7: In an open interval I containing no absorption and no translation points, there exists in I a strictly increasing function v, a $g\in C(E)$, and a choice of the independent variable x such that in I

$$Af=gD_v(f/g)'+cf$$

(D_v means derivation with respect to v). If the diffusion process preserves measure, then $g=1$ and $c=0$ so that $Af=D_vf'$. *K. Yosida* (Tokyo).

Deprit, André. **Distributions de L. Schwartz et intégrales de Cauchy.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 910–913 (1954).

Si T est une distribution sur R^2 [cf. Schwartz, Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833] et si on suppose que T est à support compact, alors

$$\pi\frac{\partial^n}{\partial z^n}T=(-1)^n n!(\text{v.p. } z^{-(n+1)})*\frac{\partial}{\partial \bar z}T.$$

L'auteur applique cette formule avec T égale à une fonction f, holomorphe dans un ouvert S borné, de frontière C régulière, f étant continue dans $S\cup C$, et nulle hors de $S\cup C$. La formule (6) de l'auteur demande à être précisée.

J. L. Lions (Nancy).

Grothendieck, A. **Sur les espaces de solutions d'une classe générale d'équations aux dérivées partielles.** J. Analyse Math. **2**, 243–280 (1953).

By "differentiable" we shall mean "differentiable of class C^∞", by "manifold" we shall mean "orientable differentiable manifold", and by "form" we shall mean "differentiable differential form." Let V be a manifold, let $\mathbf{E}(V)=\sum\mathbf{E}^p(V)$, where $\mathbf{E}^p(V)$ is the space of forms of degree p on V, and let $\mathbf{D}(V)=\sum\mathbf{D}^p(V)$, where $\mathbf{D}^p(V)$ is the subspace of $\mathbf{E}^p(V)$ which is composed of forms with compact supports. Let $\mathbf{D}'(V)$ be the space of currents (form-distributions) of all degrees on V, $\mathbf{E}'(V)$ the subspace of $\mathbf{D}'(V)$ composed of the currents with compact supports.

It is possible to define the exterior product $S\wedge T$ of two currents S, T when one of them is a form; the support of the exterior product is contained in the intersection of the supports of S and T. Moreover, if T is a current with compact support of degree n (n the dimension of V), its integral is defined. Hence a pairing between $\mathbf{D}^p(V)$ and $\mathbf{D}'^{n-p}(V)$ (or between $\mathbf{E}^p(V)$ and $\mathbf{E}'^{n-p}(V)$) is given by the formula $(\varphi, T)=\int\varphi\wedge T$. This pairing expresses the duality of $\mathbf{D}(V)$, $\mathbf{D}'(V)$ (or of $\mathbf{E}(V)$, $\mathbf{E}'(V)$).

A differential operator D on V with differentiable coefficients is a linear map D: $\mathbf{D}(V)\to\mathbf{D}(V)$ such that the support of $D\varphi$ is contained in that of φ. The differential operators in $\mathbf{D}(V)$ form an algebra, for the sum and product of two operators is again an operator. To every differential operator D there is associated a transposed operator ${}^tD$ defined uniquely by the condition $(D\varphi, T)=(\varphi, {}^tDT)$, $\varphi\in\mathbf{D}(V)$, $T\in\mathbf{D}'(V)$. More generally, $(DS, T)=(S, {}^tDT)$, where S or T is differentiable and the intersection of their supports is compact.

A kernel E is a linear (weakly continuous) map of $\mathbf{D}(V)$ into $\mathbf{D}'(V)$ defined by a current on $V\times V$ according to the rule $(E\cdot\psi, \varphi)=E(\varphi, \psi)=(\varphi\otimes\psi, E)_{V\times V}$. The transposed kernel ${}^tE$: $\mathbf{D}(V)\to\mathbf{D}'(V)$ is defined by $E(\varphi, \psi)=(\psi, {}^tE\varphi)$. A kernel E is called regular if the maps of $\mathbf{D}(V)$ into $\mathbf{D}'(V)$ defined by E and ${}^tE$ can both be prolonged into continuous maps of $\mathbf{E}'(V)$ into $\mathbf{D}'(V)$.

Let D: $\mathbf{D}(V)\to\mathbf{D}(V)$ be a differential operator. The kernel E is said to be a left inverse of D if $E\circ D$ is the identity. A right inverse of a differential operator D: $\mathbf{D}'(V)\to\mathbf{D}'(V)$ is defined similarly. If E is a left inverse of D, ${}^tE$ is a right inverse of ${}^tD$.

Let D be a differential operator on V which possesses a regular left inverse E. A solution of the equation $DS=0$ defined in the complement $U=CK$ of a compact K is said to be regular at infinity relative to the inverse kernel E if, for every $\psi\in\mathbf{D}(V)$ and every differentiable function ω with compact support which is equal to 1 is a neighborhood of $K\cup$ support ψ, $(D(\omega S), {}^tE\psi)_U=0$. It is shown, in particular, that a solution S of $DS=0$ which is defined on all of V and is regular at infinity is null.

Suppose that D is a differential operator on V which possesses a regular bilateral inverse E, and let U be an open subset of V with a compact complement. It is shown that the topological vector space $H_D(U)$ of current solutions in U of the equation $DS=0$ is the direct topological sum of the subspace of solutions regular at infinity and of the subspace of solutions which are restrictions to U of solutions defined in all of V. The component of $S\in H_D(U)$ belonging to the latter subspace is the restriction to U of the current T, solution of $DT=0$ in V, which is defined in every relatively compact open set O by $T=E[D(\omega S)]$, where ω is a differentiable function with compact support which is equal to 1 in a neighborhood of $\bar{\text{O}}\cup K$.

If V is not compact, let $\hat V$ be the compactified space obtained by adjoining to V a "point at infinity" (denoted ∞). If V is compact, define $\hat V=V$. Let D be a differential operator on V. If U is an open set of $\hat V$, let $H_D(U)$ be the space of currents defined in $U'=U\cap V$ which satisfy there the equation $DS=0$, and let $h_D(U)$ be the subspace of $H_D(U)$ composed of the differentiable solutions. These spaces depend only on $U'=U\cap V$ and, for $U\subset V$, $H_D(U)$ coincides with the notation used above. If A is an arbitrary subset of $\hat V$, let $H_D(A)$ be the space of classes of currents which are elements of $H_D(U)$, U a variable open neighborhood of A, two currents $S'\in H_D(U')$, $S''\in H_D(U'')$ being identified if and only if they define by restriction the same element of $H_D(U)$, where U is some open neighborhood of A contained in $U'\cap U''$. The space $h_D(A)$ is defined similarly.

Let U_1 and U_2 be two open sets of $\hat V$, $U_1\cup U_2=\hat V$, ∞ non-$\in U_1\cap U_2$, and let $S\in H_D(U_1)$, $T\in H_{{}^tD}(U_2)$, one of these currents being differentiable. Let ω be a function on $\hat V$, differentiable on V, equal to 1 in a neighborhood of the complement CU_2 of U_2 and equal to 0 in a neighborhood of CU_1. It is shown that $(S, T)=(D(\omega S), T)_{U_1\cap U_2}$ is independent of the choice of ω and that this scalar product does not change when S and T are restricted to smaller open sets U_1', U_2' satisfying $U_1'\cup U_2'=\hat V$. Given two non-empty sets A, B of $\hat V$ such that $A\cup B=\hat V$, ∞ non-$\in A\cap B$, and given $\tilde S\in h_D(A)$, $\tilde T\in H_{{}^tD}(B)$ (or $\tilde S\in H_D(A)$, $\tilde T\in h_{{}^tD}(B)$), define $(\tilde S, \tilde T)=(S, T)=(D(\omega S), T)_{U_1\cap U_2}$, where S, T are representatives of $\tilde S$, $\tilde T$ defined in open neighborhoods U_1 and U_2 of A and B, ∞ non-$\in U_1\cap U_2$. In particular, if A is a non-empty subset of $\hat V$ different from $\hat V$, $(\tilde S, \tilde T)$ defines a natural pairing between $H_D(A)$ and $h_{{}^tD}(CA)$ and also between $h_D(A)$ and $H_{{}^tD}(CA)$ (CA the complement of A with respect to $\hat V$).

Let $P_D(U)$ (respectively $p_D(U)$) be the space $H_D(U)$ (respectively $h_D(U)$) if ∞ non-$\in U$, the subspace composed of solutions regular at infinity if $\infty\in U$. The author establishes the following theorem of duality: If D possesses a regular left inverse E and if U is an open subset of $\hat V$, every linear continuous form on $P_D(U)$ (respectively $p_D(U)$) can be obtained from a $\tilde T\in h_{{}^tD}(CU)$ (respectively from a $\tilde T\in H_{{}^tD}(CU)$) by the above described pairing.

An appendix gives an algebraic systematization (based on the theory of de Rham) of the pairings described above. Although it is perhaps the most interesting part of the paper, lack of space makes it impossible to describe its content here.

D. C. Spencer (Paris).

Dževarševili, A. G. On the normed space of D^*-integrable functions. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 153–162 (1953). (Russian. Georgian summary)

Let D^* denote the space of functions integrable in the Denjoy-Perron sense on $(-\pi, \pi)$. For each $f \in D^*$ the author defines $\|f\| = \sup_\varphi \{|\int_{-\pi}^{\pi} f(x)\varphi(x)dx|\}$, where $\phi(x)$ is of bounded variation, $|\phi(x)| \leq 1$, $V^{\pi}_{-\pi}(\phi) \leq 1$. This norm is equivalent to one considered by A. Alexiewicz [Colloq. Math. 1, 289–293 (1948); MR 10, 717]. The author proves the following properties: The set of polynomials is dense in D^*. If $\omega(\delta) = \sup_{0<h\leq\delta} \|f(x+h)-f(x)\|$, then $\lim_{\delta\to 0} \omega(\delta) = 0$, for each $f \in D^*$. If $\{f_n\}$ is a Cauchy sequence in D^*, and if the functions $F_n(x) = \int_{-\pi}^{x} f_n(t)\,dt$ are equally-(ACG*) on $(-\pi, \pi)$, then there exists an element $f_0 \in D^*$ such that $\|f_n - f_0\| \to 0$. Some applications to the orthogonal systems of D^* are given. *M. Cotlar* (Mendoza).

Følner, Erling. On the dual spaces of the Besicovitch almost periodic spaces. Danske Vid. Selsk. Mat.-Fys. Medd. 29, no. 1, 27 pp. (1954).

For an ordinary almost periodic (a.p.) function f we define $\|f\|_{B^p}$ to be $\lim\sup_{T\to\infty} [(2T)^{-1}\int_{-T}^{T}|f|^p dx]^{1/p}$. By the usual equivalence class and completion techniques, the Banach spaces B^p, $1 \leq p < \infty$, are defined. If M is an arbitrary module of real numbers $\{\lambda\}$ for which

$$\lambda_1, \lambda_2 \in M \Rightarrow \lambda_1 - \lambda_2 \in M,$$

we call an almost periodic (a.p.) function or point in B^p an M function (point) if its non-zero Fourier coefficient indices all belong to M. The set of M points is a closed (Banach) space in B^p, say $B_M{}^p$. The paper under review gives two independent proofs of the fact that the conjugate space of $B_M{}^p$ is $B_M{}^q$ if $1 < p < \infty$, $1/p + 1/q = 1$. The conjugate space of $B_M{}^1$ is $B_M{}^\infty$ (obvious notation). For methods, the paper provides its own review: ". . . The first method . . . uses only the ordinary theory of generalized a.p. functions. It is based on previous results of R. Doss [Ann. of Math. (2) 46, 196–219 (1945); Amer. J. Math. 72, 81–92 (1950); MR 6, 265; 11, 659] and is an extension of the method used by Doss. . . . The second method . . . consists in the establishment of a close correspondence [linear isometry] between the points of $B_M{}^p$ and the points of $[L^p(R_M{}^1)]$, where $R_M{}^1$ is the Bohr compactification of the real axes by all ordinary a.p.-M functions. *B. Gelbaum.*

Nikodým, Otton Martin. Limit-representation of linear, even discontinuous, linear functionals in Hilbert spaces. Rend. Sem. Mat. Univ. Padova 23, 290–298 (1954).

This note proves the following theorem: Let W be a linear topological space such that for every finite-dimensional linear subspace E there is a complementary linear subset E_1 such that for every linear functional f defined on E there is a linear continuous extension f_1 of f defined on all W and vanishing on E_1; let V be a linear subset of W and let f be a not-necessarily-continuous, linear functional defined on V; then there exists a directed system (D, R) and for each d in D a linear continuous functional f_d on W such that for each x in V the Moore-Smith limit

$$\lim_{d \in (D,R)} f_d(x) = f(x);$$

while the limit fails to exist for each x not in V. The author is interested in the choice of Hilbert space for W but gives the general result above; it should be observed that the condition on W is equivalent to the condition that W^*, the set of linear continuous functionals on W, is total over L (that is, that if $x \in W$ and $x \neq 0$, then there is f in W^* with $f(x) \neq 0$) and to the condition that each finite-dimensional subspace of W is the range of a linear continuous idempotent operator on W. *M. M. Day* (Urbana, Ill.).

Putnam, C. R. On the spectra of commutators. Proc. Amer. Math. Soc. 5, 929–931 (1954).

If A and B are bounded operators on a Hilbert space and $AC = CA$, where $C = AB - BA$, then there exists a sequence of operators C_n such that $\|C - C_n\| \to 0$ and the spectral radius of $C_n \to 0$ as $n \to \infty$. If, in addition, $BC = CB$, then the spectral radius of C is zero. As a consequence, if $(AA^* - A^*A)A = A(AA^* - A^*A)$, then A is normal. *S. Sherman* (Philadelphia, Pa.).

Krasnosel'skiĭ, M. A. On the stability of critical values of even functionals on the sphere. Dokl. Akad. Nauk SSSR (N.S.) 97, 957–959 (1954). (Russian)

Let S be the sphere $\|\varphi\| = 1$, T the region $\|\varphi\| \leq 1$ in the Hilbert space H; let the functional $F_0(\varphi)$ be weakly continuous and uniformly differentiable in T, even and positive on S; and let its gradient $\Gamma_0\varphi$ be $\neq 0$ in those points $\varphi \in T$, where $F_0(\varphi) > 0$. The point $\varphi \in S$ is a critical point of $F_0(\varphi)$ on S if $\Gamma_0\varphi = (\Gamma_0\varphi, \varphi)\varphi$.

Let M_k be the class of sets $E \subset S$ each of which is the odd continuous image in S of the k-dimensional sphere, and let $c_k = \sup_{E \in M_k} \{\inf_{\varphi \in E} F_0(\varphi)\}$ $(k = 1, 2, \cdots)$. The author asserts that each c_k is positive and a critical value of $F_0(\varphi)$ on S and that there are infinitely many different numbers among the c_k (Theorem 1). He then presents still another class of positive critical values of $F_0(\varphi)$ on S, which also contains infinitely many different numbers. What the relation of these critical values is to those found by Lyusternik [Izv. Akad. Nauk SSSR. Ser. Mat. 3, 257–264 (1939)] is not discussed. Finally it is asserted that for every n there is a $\delta > 0$ such that the perturbed functional $F(\varphi) = F_0(\varphi) + G(\varphi)$ has at least n different critical values on S if the weakly continuous and uniformly differentiable $G(\varphi)$ and its gradient $\Gamma_1(\varphi)$ satisfy the inequality

$$\sup_{\varphi \in S} |G(\varphi)| + \sup_{\varphi \in S} \|\Gamma_1\varphi\| < \delta$$

(Theorem 5). No proofs are given. *M. Golomb.*

Lippmann, Horst. Differentialoperatoren im Hilbertraum. Math. Nachr. 12, 9–28 (1954).

Operators defined on a separable Hilbert Space H and isomorphic to d/dx on $L(-\pi, \pi)$, or to its contractions are discussed. Let $\{e_n\}$ $(n = 0, \pm 1, \cdots)$ be a complete normal orthogonal set. Let C be the least closed linear operator such that $Ce_n = ne_n$ for all n, and D the least closed linear extension of C such that $D(\sum_{n\neq 0} (-1)^n e_n/n) = -ie_0$. Then D is isomorphic to the operator d/dx defined in $L^2(-\pi, \pi)$ for all functions with differential coefficients in L^2; C is a normal contraction of D. The nature of bounded and of certain classes of unbounded operators which commute with D or C is discussed. *J. L. B. Cooper* (Cardiff).

Mihlin, S. G. Some theorems of the theory of operators and their application in the theory of elastic shells. Dokl. Akad. Nauk SSSR (N.S.) 84, 909–912 (1952). (Russian)

Let H be a Hilbert space and A be a positive, self-adjoint operator on H [for the definitions used see S. G. Mihlin, Direct methods in mathematical physics, Gostehizdat,

Moscow-Leningrad, 1950; MR 16, 41], with domain of definition $D(A)$. Let H_A denote the Hilbert space obtained by completing $D(A)$ using the metric defined by $[u, v]=(Au, v)$. The theorems referred to in the title are the following. Theorem 1: The domain of definition $D(\sqrt{A})$ of $\sqrt{A}$ is the intersection of H_A and H. Theorem 2: If A and B are positive, self-adjoint, on H, and (1) $D(\sqrt{B})\supset D(\sqrt{A})$, (2) there exist constants δ' and δ'' such that

$$\delta'(Au, u)\leqq (Bu, u)-(Au, u)\leqq \delta''(Au, u),$$

then the spectrum of the self-adjoint extension of the operator $(\sqrt{A^{-1}}\sqrt{B})(\sqrt{B}\sqrt{A^{-1}})-J$, where J is the identity operator, lies in the interval (δ', δ''). These theorems are used to estimate the "error" in replacing the solution of the equation $Bu=f$, where f is in both $D(B^{-1})$ and $D(A^{-1})$, by the solution of the equation $Au_0=f$. The above considerations are applied to what V. Z. Vlasov [General theory of shells, Gostehizdat, Moscow-Leningrad, 1949; MR 11, 627] calls "the equations of the technical theory of shells", upon first neglecting the terms which are proportional to the square of the principal curvatures of the middle surface of the shell. *J. B. Diaz* (College Park, Md.).

Colmez, Jean. Définition de certains opérateurs différentiels dans un espace de Hilbert de fonctions de carré sommable. C. R. Acad. Sci. Paris **240**, 37–39 (1955).

The author studies domains in Hilbert space to which the hydrogen-atom hamiltonian operator $\Delta+(1/r)$, or powers thereof, may be applied, and related questions.

I. E. Segal (Chicago, Ill.).

Lyance, V. E. On differential equations in unitary space. Dopovidi Akad. Nauk Ukrain. RSR **1952**, 258–262 (1952). (Ukrainian. Russian summary)

The differential equation considered is

$$\text{(a)} \qquad du/dt=q(t, A)+f(t),$$

where $u(t)$, $f(t)$ are functions of the real variable t with values in a unitary vector space H, A is a self-adjoint (not necessarily bounded) operator in H, and $q(t, \alpha)$ a complex-valued function continuous in t and measurable in α with respect to the spectral measure $E(\Delta)$ of the operator A. It is shown that (a) has no more than one solution satisfying the Cauchy condition (b) $u(t_0)=u_0$, $u_0\in H$. If f, u_0 satisfy a certain condition, then the problem (a), (b) has the solution

$$u(t)=Q(t, t_0, A)u_0+\int_{t_0}^{t}Q(t, \tau, A)f(\tau)d\tau,$$

where $Q(t, t_0, \alpha)=\exp\int_{t_0}^{t}q(\tau, \alpha)d\tau$. The author also gives a necessary condition on u_0 for the existence of a solution of (a), (b) with $f=0$. *M. Golomb* (Lafayette, Ind.).

Szökefalvi-Nagy, Béla. On a moment problem for self-adjoint operators. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 163–171 (1954). (Hungarian)

Hungarian version of Acta Math. Acad. Sci. Hungar. **3**, 285–293 (1952); MR **14**, 1096.

Pallu de La Barrière, Robert. Sur les algèbres d'opérateurs dans les espaces hilbertiens. Bull. Soc. Math. France **82**, 1–52 (1954).

In this paper the author generalizes certain results of von Neumann and Murray from the case in which the algebra of operators is a factor in a separable Hilbert space to the general case in which the Hilbert space need not be separable and the algebra need not be a factor. The results in question are those having to do with the numerical invariant C and its role in determining the spatial type of an algebra of operators. The author adopts the "global" point of view inaugurated by Dixmier and further developed by Kaplansky. He does not make use of the canonical decomposition as a continuous sum of factors which indeed is applicable only in the separable case.

In the general case C turns out to be a function defined on the spectrum of the center of the algebra and taking on values which may be positive real numbers, infinite cardinals or formal ratios of infinite cardinals. [For operator algebras of finite class such a generalization of C was defined earlier by Kaplansky, C. R. Acad. Sci. Paris **231**, 485–486 (1950); MR **12**, 186.] In the present paper only those algebras are excluded which have a nonzero type III component.

The paper is divided into three chapters. Ch. I contains various preliminary material leading up to the definition of the function C. Some of this material is of interest in itself. Ch. II is concerned with the special case in which C is everywhere greater than or equal to one and the algebra is of finite class. The von Neumann-Murray result for factors according to which $C\geqq 1$ implies the existence of an element x such that $\operatorname{Trace} A=(A(x), x)$ is generalized and the generalized result used to investigate the properties of numerical traces. A canonical isomorphism of the algebra with its commutator is established when C is identically one. The main result of Ch. III asserts that under certain conditions an algebraic isomorphism between two algebras which preserves C is necessarily a spatial isomorphism. A number of related results are given. In particular, a number of conditions are given under which an algebraic isomorphism has various continuity properties.

The principal results were summarized earlier in two notes [ibid. **232**, 1994–1995 (1951); **234**, 795–797 (1952); MR 12, 837; 13, 756]. An investigation of the same question has been carried out independently by E. L. Griffin [Trans. Amer. Math. Soc. **75**, 471–504 (1953); MR **15**, 539].

G. W. Mackey (Cambridge, Mass.).

Myers, S. B. Algebras of differentiable functions. Proc. Amer. Math. Soc. **5**, 917–922 (1954).

It is shown that the structure of a compact differentiable manifold of class C^r is determined by the algebra $C^r(M)$ of all real functions of class C^r of M. If $C^r(M)$, $C^r(M')$ are algebraically isomorphic, then there is a differentiable homeomorphism, which, with its inverse, is of class C^r, of M onto M'. If M has a metric tensor g_{ij} of class $r-1$, and a norm in $C^r(M)$ is determined by

$$\|f\|=\max|f(x)|+\max\left(g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\right)^{1/2},$$

then $C_1(M)$ is a real commutative normed algebra with unit, and for $r=1$ is a Banach algebra. If $C^r(M)$ and $C^r(M')$ are equivalent as normed algebras, then there is an isometry of class C^r of M on M'. These theorems are proved by means of a set of lemmas which are in effect separation theorems for functions in the algebras. *J. L. B. Cooper.*

Yoshizawa, Hisaaki. A proof of the Plancherel theorem. Proc. Japan Acad. **30**, 276–281 (1954).

The author considers the Plancherel theorem for commutative *-algebras, essentially as formulated by Godement [Ann. of Math. (2) **53**, 68–124 (1951); MR **12**, 421]. The proof given here depends upon a preliminary reduction to

the case in which the underlying *-algebra is a rather special kind of algebra of operators. This case is then handled with the aid of an approximate identity. *G. W. Mackey.*

Mařík, Jan. Extreme points of the unit sphere in the space of functionals on a given partially ordered space. Časopis Pěst. Mat. **79**, 3–40 (1954). (Czech)

A K-lineal is a linear space Y over the real number field R admitting a lattice ordering compatible with the algebraic operations in Y and the order in R. A normed K-lineal is a K-lineal admitting a norm $\|\cdot\|$ such that $0\leqq a\leqq b$ imply $\|a\|\leqq\|b\|$ and $\|a\|=\|(a\vee 0)+((-a)\vee 0)\|$. Theorem A. Let Y be a normed K-lineal such that

$$\|a\vee b\|=\max(\|a\|,\|b\|)$$

if $a\wedge b\geqq 0$. Then the extreme points of the unit cell in the conjugate space Y^* of Y are exactly the $f\in Y^*$ such that $\|f\|=1$ and $f(a)f(b)=0$ if $a\wedge b=0$. This fact is closely related to a theorem of Kakutani [Ann. of Math. (2) **42**, 994–1024 (1941); MR **3**, 205], and can be inferred easily from the proof of Kakutani's theorem. Kakutani's theorem can also be obtained from Theorem A.

A K-lineal Y is said to be a K-ring if Y is a commutative algebra over R with unit j having the property that $(a\vee b)c=(ac)\vee(bc)$ for all $a, b, c\in Y$ such that $c\geqq 0$. Let $B(Y)$ denote the set of all linear functionals on the K-lineal Y that are differences of non-negative functionals. It is known that $B(Y)$ is a K-lineal. If Y is a K-ring, then $\|f\|=((f\vee 0)+((-f)\vee 0))(j)$ is a norm for $B(Y)$. Theorem B. Let Y be a K-ring. Then the extreme points of the unit cell in $B(Y)$ are exactly the functionals $\pm h$, where h is an algebra-homomorphism of Y onto R. This generalizes a theorem of the reviewer [Fund. Math. **37**, 161–189 (1950), Theorem 27; MR **13**, 147].

The paper presents in addition a very readable account of the elementary theory of K-lineals and K-rings, along with a number of subsidiary new results. *E. Hewitt.*

Calculus of Variations

Velte, Waldemar. Zur Variationsrechnung mehrfacher Integrale in Parameterdarstellung. Mitt. Math. Sem. Giessen no. **45**, i+33 pp. (1953).

Let $t=[t_1,\cdots,t_\mu]$, $x=[x_1,\cdots,x_n]$ $(\mu<n)$; let $p=\{p_{(i)}\}$ denote the set of all $\mu\times\mu$ minor determinants $p_{(i)}$ of the matrix $(\partial x_i/\partial t_\alpha)$, where $(i)=i_1,\cdots,i_\mu$ with $i_1<i_2<\cdots<i_\mu$. The $p_{(i)}$ satisfy a certain set of relations which express that p belongs to the so-called Grassman manifold. Let $f(x,p)$ be a sufficiently-many-times differentiable function with the properties: (a) $f(x,cp)=cf(x,p)$ for $c>0$, (b) $f>0$ unless all $p_{(i)}=0$, (c) $\{\partial f/\partial p_{(i)}\}$ belongs to the Grassman manifold whenever p does. Following methods of Carathéodory [Variationsrechnung und partielle Differentialgleichungen erster Ordnung, Teubner, Leipzig-Berlin, 1935] and Boerner [Math. Z. **46**, 720–742 (1940); MR **2**, 225], the author considers the variational problem $\int f(x,p)dt=$minimum for μ-dimensional parametric surfaces in n-space with boundary fixed or variable on a manifold. The notion of geodesic field is defined, and it is shown that a positive regular surface element embeddable in a geodesic field provides a weak relative minimum. It is also proved that each sufficiently small portion of a four times continuously differentiable extremal $x(t)$ can be embedded in a geodesic field, provided $\{p_{(i)}(t)\}\neq\{0\}$ for all t. *W. H. Fleming.*

Velte, Waldemar. Zur Variationsrechnung mehrfacher Integrale. Math. Z. **60**, 367–383 (1954).

The author adapts to parametric m-ple integrals the sufficiency theory developed by Carathéodory in the nonparametric case and proves that a regular extremal can be embedded in a geodesic field. The process of adaptation is by no means trivial, and indeed the author's original account [see the preceding review] contained a restriction on the integrand which he has now eliminated with the help of an argument due to H. Kneser [Bull. Soc. Math. Grèce **20**, 101–103 (1940); MR **2**, 60]. The paper contains some implicit assumptions which are not very clear: for instance, the topological nature of the parameter domain is not mentioned, and it is not stated whether the surfaces concerned are oriented. It appears to the reviewer that a suitable sprinkling of definitions would add both clarity and generality to the author's otherwise interesting presentation. *L. C. Young* (Madison, Wis.).

Bullen, K. E. Conversion of variation problems into isoperimetrical problems. Math. Gaz. **38**, 249–252 (1954).

De Donder, Th. Sur les multiplicateurs indéterminés d'Euler-Lagrange. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 877–879 (1954).

This note gives formally a Lagrange-multiplier rule for multiple-integral variational problems with partial differential equations of arbitrary order as side conditions. There is no trace of a proof, and no mention is made of questions of normality. *L. M. Graves* (Chicago, Ill.).

Manaresi, Fabio. Un problema di autovalori. Rend. Sem. Mat. Univ. Padova **23**, 343–351 (1954).

Variational methods are used to establish the existence of proper values and proper functions for the boundary problem involving the Euler-Lagrange equation for the problem of extremizing

$$I[u]=\int\int_R[\theta u^2_{xy}+ru^2_x+2su_xu_y+tu_y^2+pu^2]\,dxdy$$

in a class Γ_0 of functions $u(x,y)$ satisfying on the rectangular domain R: $a_1\leqq x\leqq a_2$, $b_1\leqq y\leqq b_2$ suitable differentiability conditions, vanishing on the boundary of R, and providing a fixed value to the integral

$$(*)\qquad \int\int_R[2\sigma u_{xy}+2\alpha u_x+2\beta u_y+qu]u\,dx\,dy.$$

The specific hypotheses on the coefficients are such that $I[u]$ is positive definite on Γ_0, while the integral (*) may be indefinite. The results of this paper are direct extensions of those of a recent paper of the author [same Rend. **23**, 163–213 (1954); MR **16**, 138]. *W. T. Reid.*

Theory of Probability

***Fisz, Marek. Rachunek prawdopodobieństwa i statystyka matematyczna.** [**The calculus of probabilities and mathematical statistics.**] Państwowe Wydawnictwo Naukowe, Warszawa, 1954. 374 pp. zł. 31.50.

This book is, to the reviewer's knowledge, the first comprehensive presentation of the mathematical theory of statistics in Polish and constitutes a valuable contribution to

the scientific literature in this language. It follows a by now well established pattern: About one-half of the 350 pages is devoted to an exposition of those parts of calculus of probabilities which are of immediate use in mathematical statistics, while the second half of the book deals with statistical theory.

Probability is introduced axiomatically, following essentially Kolmogorov's presentation. From then on only random variables with discrete distributions and random variables which have a probability density are considered. While generally an attempt is made to keep up with the present state of the theory, various statements have slipped in which were taken over from conventional books on statistics. For example kurtosis is discussed as a measure of peakedness, without warning on possible pitfalls. Fundamental properties of characteristic functions are introduced, mostly without proof, and used to derive theorems in probability, in particular, limit theorems. Most of the usual discrete and continuous probability distributions are discussed in considerable detail. Cauchy's distribution is not fully used as the source of counter-examples which it can provide; e.g., it is not pointed out that the arithmetic mean of observations from that population has the same distribution as a single observation.

Part Two, dealing with mathematical statistics, consists of chapters with the following headings: Exact distributions of statistics. Asymptotic distributions of statistics. Significance tests. Theory of estimation. Methods and design of sampling. Outline of analysis of variance. General theory of tests. Elements of sequential analysis. The chapters dealing with general theory present and discuss contemporary concepts pertaining to estimation and testing hypotheses in the form given by Neyman and Pearson. A number of proofs are barely indicated or entirely omitted. The discussion of generally accepted statistical techniques is quite complete and fairly detailed. A notable exception occurs in presenting distribution-free statistics: the chi-square test, Kolmogorov's and Smirnov's statistics, and quite a bit of order statistics are discussed, but no mention is made of distribution-free tolerance limits, or tests based on runs. Most of the illustrations throughout the book are from the fields of industrial and agricultural statistics; applications to genetics are not mentioned.

Z. W. Birnbaum (Seattle, Wash.).

Helphen, Etienne. Sur l'analyse intrinsèque d'une distribution. C. R. Acad. Sci. Paris **239**, 1265–1266 (1954).

Fabián, Václav. A note on the conditional expectations. Čehoslovack. Mat. Ž. **4**(79), 187–191 (1954). (Russian summary)

Let ξ, η be random variables. Then, intuitively, the conditional expectation of $h(\xi, \eta)$, for $\eta = y$, where y is a fixed number, is equal to the conditional expectation of $h(\xi, y)$, for $\xi = y$. The author sets up a formalism too technical to reproduce here in whose terms this and related statements can be proved, under appropriate restrictive hypotheses. These hypotheses are satisfied, for example, in the above assertion, if ξ is the coordinate variable in euclidean n space, if the given probability measure is a distribution on the Borel sets of this space, if η is a random variable on the same space, and if h is a Baire function such that the expectation of $h(\xi, \eta)$ exists.

J. L. Doob (Urbana, Ill.).

Krishna Iyer, P. V., and Kapur, M. N. Probability distributions arising from points on a line. Biometrika **41**, 553–554 (1954).

Let X_i $(1 \leq i \leq n)$ be independent and $P\{X_i = A_j\} = p_j$ $(j = 1, 2, 3)$. The authors compute the first four cumulants of the number of adjoining links (X_i, X_{i+1}) of the form (A_r, A_s) with $r \neq s$. *J. Kiefer* (Ithaca, N. Y.).

Good, I. J. The joint distribution for the sizes of the generations in a cascade process. Proc. Cambridge Philos. Soc. **51**, 240–242 (1955).

Consider a branching process with s types of individuals. Let $\mathbf{F}_g(\mathbf{x}) = (F_{g1}(x_1, \cdots, x_s), \cdots, F_{gs}(x_1, \cdots, x_s))$ be the generating function for the progeny of an individual in the gth generation. (Clarendon type denotes s-dimensional vectors; F_{gt} is the generating function for the progeny of an individual in generation g of type t; $\mathbf{a} \wedge \mathbf{b}$ denotes $(a_1b_1, \cdots, a_sb_s)$; $\mathbf{a}^{\mathbf{b}} = a_1^{b_1}a_2^{b_2} \cdots a_s^{b_s}$.) The joint generating function for individuals in the 1st, $\cdots$, nth generation, starting with the configuration $\mathbf{k}_0$ in the zero-th generation, is shown to be $\mathbf{F}_0(\mathbf{x}_1 \wedge \mathbf{F}_1(\mathbf{x}_2 \wedge \cdots \wedge \mathbf{F}_{n-1}(\mathbf{x}_n)) \cdots))^{\mathbf{k}_0}$. A number of formulas, new and old, concerning branching processes are shown to follow as corollaries. *T. E. Harris.*

Steriotis, P. J. Probability of transvariation in the regular distributions of Gauss. Bull. Soc. Math. Grèce **28**, 1–36 (1954). (Greek. English summary)

Bertaut, Félix. Sur la probabilité de valeurs de fonctions. Application à la cristallographie. C. R. Acad. Sci. Paris **240**, 152–154 (1955).

Bertaut, Félix. Fonctions de répartition. Paramètres les plus probables. Application à la détermination directe des structures atomiques en cristallographie. C. R. Acad. Sci. Paris **240**, 272–274 (1955).

Let $(x_i : 0 < i \leq n)$ be independent random variables, uniformly distributed on $(0, 1)$. The joint probability density of a finite collection of random variables on the sample space of the $(x_i : 0 < i \leq n)$ is expressed as the expectation of a product of Dirac functions. The manipulations are completely formal, and the applications without mathematical interest. *H. P. McKean, Jr.* (Princeton, N. J.).

Skorohod, A. V. Asymptotic formulas for stable distribution laws. Dokl. Akad. Nauk SSSR (N.S.) **98**, 731–734 (1954). (Russian)

Asymptotic expressions with error estimates as $x \to \pm\infty$ and $x \to 0$ of all stable density functions $p(x, \alpha, \beta)$ are completely tabulated, where α and β have the usual meanings in the canonical form of the characteristic function. The known results are due to Pollard, Bergström and Linnik. The new expressions correspond to the following cases: (1) $\alpha = 1$, $x \to +\infty$, (2) $\alpha = 1$, $x \to -\infty$, (3) $\alpha > 1$, $\beta = \mp 1$, $x \to \pm\infty$. *K. L. Chung* (Syracuse, N. Y.).

Zolotarev, V. M. Expression of the density of a stable distribution with exponent α greater than one by means of a density with exponent $1/\alpha$. Dokl. Akad. Nauk SSSR (N.S.) **98**, 735–738 (1954). (Russian)

Let $p(x, \alpha, \beta)$ be as in the preceding review and let $\alpha > 1$, $x > 0$. The expression in the title is of the form

$$axp(-ax, \alpha, \beta) = a'x^{-\alpha}p(a'x^{-\alpha}, \alpha^{-1}, \beta'),$$

where the coefficients a, a' and β' depend on α and β in a complicated way. If $\alpha < 1$ is rational a complicated differ-

ential equation is obtained for the function p which generalizes a recent result of Linnik [same Dokl. (N.S.) 94, 619–621 (1954); MR 16, 378]. *K. L. Chung.*

Cramér, Harald. On some questions connected with mathematical risk. Univ. California Publ. Statist. 2, 99–123 (1954).

A mathematically precise and unified exposition of the theory of collective risk (for an insurance company), with rigorous proofs of some of its principal results and references to the principal literature. *J. Wolfowitz* (Ithaca, N. Y.).

Gnedenko, B. V. On a local limit theorem for identically distributed independent summands. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 3, 287–293 (1954). (Russian)

The author proves two theorems, of which the first is already given in Gnedenko and Kolmogoroff "Limit distributions for sums of independent random variables" [Gostehizdat, Moscow-Leningrad, 1949, §50, p. 252; English translation published by Addison-Wesley, Cambridge, Mass., 1954, §50, p. 236; MR 12, 839; 16, 52] and the other is as follows: Let $x_1, x_2, \cdots$ be independent, identically distributed chance variables, let $S_n = x_1 + \cdots + x_n$, and suppose that, for some $n = n_0$ (say), the density function $p_{n_0}(x)$ of S_{n_0} exists. (Then of course the density function $p_n(x)$ of S_n exists for all $n > n_0$.) Let $p(x)$ be the density function of a stable law. Then, in order that

$$\sup_x |B_n p_n(B_n x + A_n) - p(x)| \to 0 \quad \text{as} \quad n \to \infty$$

it is necessary and sufficient that 1) for some integer $n = n_1$ (say), $p_{n_1}(x)$ is bounded, and

$$\text{2)} \qquad P\left\{\frac{S_n - A_n}{B_n} < x\right\} \to \int_{-\infty}^{x} p(y)dy.$$

J. Wolfowitz (Ithaca, N. Y.).

Prohorov, Yu. V. On a local limit theorem for lattice distributions. Dokl. Akad. Nauk SSSR (N.S.) 98, 535–538 (1954). (Russian)

Let $s_n = \sum_1^n \xi_j$, where $\xi_1, \xi_2, \cdots$ are mutually independent uniformly bounded integer-valued random variables. Let $A_n = E\{s_n\}$, $B_n^2 = \text{var. } s_n \to \infty$, and suppose that, as a matter of notation, $\Pr\{\xi_n = 0\} = \max_j \Pr\{\xi_n = j\}$. The author proves that a necessary and sufficient condition that

$$(*) \quad \Pr\{s_n = m\} = (2\pi)^{-1/2} B_n^{-1} \exp\{-(m - A_n)^2/2B_n^2\} + o(B_n^{-1}) \quad (n \to \infty),$$

uniformly for $-\infty < m < \infty$, and that (*) remain true after any change of a finite number of ξ_j's is that 1 is the highest common factor of the values of j with $\sum_{n=1}^{\infty} \Pr\{\xi_n = j\} = \infty$. *J. L. Doob* (Urbana, Ill.).

Loève, Michel. Relations entre lois limites. C. R. Acad. Sci. Paris 239, 1585–1587 (1954).

A sequence of independent, infinitesimal random variables converges in distribution to an infinitely divisible (i.d.) one if and only if the sequences of their maxima and minima (in each row of the double-indexed scheme) converge in distribution to certain simple distributions determined by the canonical representation of the i.d. distribution. Similarly for certain functions of the random variables considered recently by Bochner [Proc. Nat. Acad. Sci. U. S. A. 40, 699–703 (1954); MR 16, 379]. *K. L. Chung.*

Marsaglia, George. Iterated limits and the central limit theorem for dependent variables. Proc. Amer. Math. Soc. 5, 987–991 (1954).

The author gives some new results concerning iterated limits and iterated limits-in-probability and applies them to obtain strikingly simple proofs of some central limit theorems for m-dependent variables. Let

$$\text{L } a_{ij} = a \text{ mean } \lim_{i\to\infty} \limsup_{j\to\infty} |a_{ij} - a| = 0,$$

and let PL $f_{ij} = f$ (where f_{ij} and f are random variables) mean that for every $\epsilon > 0$, L pr $(|f_{ij} - f| > \epsilon) = 0$. Let h_{ij}, g_{ij} $(i, j = 1, 2, \cdots)$ be random variables, let

$$\text{L pr } (g_{ij} \leqq x) = G(x)$$

at each continuity point of G and let PL $h_{ij} = 0$. Then (Theorem 1) L pr $(g_{ij} + h_{ij} \leqq x) = G(x)$. If further L $a_{ij} = a > 0$ and G is continuous at x/a, then (Theorem 2)

$$\text{L pr } (a_{ij} g_{ij} \leqq x) = G(x/a).$$

The applications include proofs of Diananda's improved form of a theorem of Hoeffding and Robbins ("the central limit theorem" for stationary m-dependent sequences with zero mean and finite variance) and a similar theorem of the Liapounoff type involving a condition on the absolute moment of order α $(\alpha > 2)$. [For the earlier work see W. Hoeffding and H. E. Robbins, Duke Math. J. 15, 773–780 (1948); MR 10, 200; and P. H. Diananda, Proc. Cambridge Philos. Soc. 49, 239–246 (1953); MR 14, 771.] *D. G. Kendall* (Oxford).

Siraždinov, S. H. Limit theorems for stationary Markov chains with continuous time. Dokl. Akad. Nauk SSSR (N.S.) 98, 905–908 (1954). (Russian)

The author considers a continuous-parameter Markov chain with stationary transition probabilities, and states $e_1, \cdots, e_s$. It is supposed that the probability of a transition from e_i to e_j in time t is positive for all i, j, and near 1 for $i = j$ when t is near 0. Let $\xi(t)$ be the vector whose jth component is the total time (before time t) that the system is in state e_j. The author gives without proof an expression for the distribution of $(\xi(t) - t\omega)/\sqrt{t}$, where ω is an explicitly given vector. This expression depends on the positive integer k, and the error term is $O(t^{-(k+1)/2})$. If $h_1, \cdots, h_s$ are numbers, at least two of which are unequal, let $u(t) = h_j$ if the chain is in state e_j, at time t. The author gives an expression for the distribution of $\int_0^t u(s)\, ds$ (after normalization), with an error term as above. This result generalizes results of Doeblin [Bull. Sci. Math. (2) 62, 21–32 (1938)], who only found the asymptotic limiting distribution. *J. L. Doob.*

Patankar, V. N. A note on recurrent events. Proc. Cambridge Philos. Soc. 51, 96–102 (1955).

In a closed aperiodic Markoff chain with finitely many states, let $N_r(i)$ be the number of time-points in the first r which are spent in state i. The author computes the asymptotic covariance between $N_r(i)$ and $N_r(j)$ as $r \to \infty$. *J. Kiefer* (Ithaca, N. Y.).

Yaglom, A. M. Effective solutions of linear approximation problems for processes with random stationary nth increments. Dokl. Akad. Nauk SSSR (N.S.) 98, 189–192 (1954). (Russian)

Among the results stated without proof are the following. Let a continuous parameter stochastic process be given, with stationary (wide sense) nth increments. [See Yaglom

and Pinsker, same Dokl. (N.S.) **90**, 731–734 (1953); **94**, 385–388 (1954); MR **15**, 238, 806.] The author gives the explicit evaluation for the linear prediction s units ahead, when the process has spectral density of the form

$$C\left|\prod_1^M(\lambda-\beta_k)\prod_1^N(\lambda-\alpha_j)^{-1}\right|^2(1+\lambda^{2n}),$$

where $C>0$, $M<N+n$, $\operatorname{Im}\alpha_j>0$, $\operatorname{Im}\beta_k>0$. The explicit solution of the filter problem is given in the same case. Corresponding results are stated when the process is only supposed known for a finite interval of the past.

J. L. Doob (Urbana, Ill.).

Sarymsakov, T. A. On the theory of stationary stochastic processes without aftereffect. (Differential equations for characteristic functions.) Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **5**, 61–69 (1949). (Russian)

Let $u(t)=x_k$ when a Markov chain is in state k, and consider the random variable $w=\int_{t_0}^t[u(s)-\bar{u}]\,ds$, where $\bar{u}$ is a constant. Let $f_k(t_0,t)$ be the characteristic function of w under the hypothesis that $u(t_0+t+0)=x_k$. Under various regularity hypotheses, the author finds a system of linear differential equations satisfied by the f_k's, as functions of t. Similarly, if the process is a diffusion process, and if $f(x,t_0,t)$ is the characteristic function of w under the hypothesis that $u(t)=x$, a parabolic partial differential equation is found for f, involving the variables t, x, and the diffusion coefficients of the given process. The reviewer was unable to follow the derivation. *J. L. Doob* (Urbana, Ill.).

Mihoc, Gh. Markov chains. Gaz. Mat. Fiz. Ser. A. **6**, 438–446, 485–493 (1954). (Romanian)

Expository paper.

Lévy, Paul. Le mouvement brownien à $n=2p+1$ paramètres. C. R. Acad. Sci. Paris **239**, 1181–1183 (1954).

To each point A of n-space corresponds the Gaussian random variable $X(A)$, and $E\{[X(A)-X(B)]^2\}$ is the distance AB. Let $M(t)$ be the average of $X(A)-X(0)$ on the spherical surface with center 0, radius t. Then it is stated that, for $n=2p+1$, $2p+2$, the sample functions of the $M(t)$ stochastic process have p derivatives, and that the random component of $\delta M(t)$, when $M(s)$ is known for $s<t$, is of the form $c_n\xi t^{1/2}(dt/t)^{n/2}$, where ξ is a normalized Gaussian random variable. Proofs and further details are given for $n=3,5$. The corresponding result for $n=\infty$ (when the n-space is replaced by Hilbert space) is also indicated. If $t=e^{2u}$, and $M(t)=e^uU(x)$, the $U(x)$ stochastic process is Gaussian and stationary. The covariance function of this process is given explicitly for $n=\infty$. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Le mouvement brownien à $n=2p+1$ paramètres. II. C. R. Acad. Sci. Paris **239**, 1584–1585 (1954).

The author continues, and in part corrects, his earlier results given in the paper reviewed above. If $X(A)$ is the random variable corresponding to the point A, and if $M(t)$ is the average of $X(A)-X(0)$ on the spherical surface with center 0, radius t, the $M(t)$ process has the property that its sample functions have p derivatives, which, together with $M(t)$ itself, determine a $(p+1)$-dimensional Markov process satisfying a simple stochastic differential equation. The covariance function of the $M(t)$ process then has a simple form. *J. L. Doob* (Urbana, Ill.).

Derman, Cyrus. Ergodic property of the Brownian motion process. Proc. Nat. Acad. Sci. U. S. A. **40**, 1155–1158 (1954).

The author proves that, if $\{X(t),0\leq t<\infty\}$ is a separable Brownian motion process, and if f, g are any two Baire functions, with finite integrals $\bar{f}$, $\bar{g}\neq 0$ respectively, over $(-\infty,\infty)$, then

$$\lim_{T\to\infty}\int_0^T f[X(t)]dt\Big/\int_0^T g[X(t)]dt=\frac{\bar{f}}{\bar{g}},$$

with probability 1. Kallianpur and Robbins [same Proc. **39**, 525–533 (1953); MR **15**, 44] proved this theorem for f, g bounded and with stochastic convergence instead of probability 1 convergence. The result can be treated as a special case of the ergodic theorem with infinite-valued measures, but the author gives a more elementary direct proof.

J. L. Doob (Urbana, Ill.).

Pinsker, M. S. The quantity of information about a Gaussian random stationary process, contained in a second process connected with it in a stationary manner. Dokl. Akad. Nauk SSSR (N.S.) **99**, 213–216 (1954). (Russian)

Let $\{\eta_n,-\infty<n<\infty\}$ be a stationary integral-parameter Gaussian stochastic process with zero means and spectral distribution having the derivative $f_{\eta\eta}$. Then it is shown that the amount of information per step of the process is given by the formula

$$\log(4\pi^2e)^{1/2}+\frac{1}{4\pi}\int_{-\pi}^{\pi}\log f_{\eta\eta}(\lambda)d\lambda,$$

(so that this amount is finite if and only if the process is nonsingular, that is, non-deterministic). If $\{(\xi_n,\eta_n),-\infty<n<\infty\}$ is a stationary integral-parameter vector Gaussian process with zero means, and if either the ξ_n or η_n process is nonsingular, then, with the obvious extension of the above notation, the amount of information per step of the one process relative to the other is given by the formula

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\frac{f_{\xi\xi}(\lambda)f_{\eta\eta}(\lambda)}{f_{\xi\xi}(\lambda)f_{\eta\eta}(\lambda)-|f_{\xi\eta}(\lambda)|^2}d\lambda.$$

In each case the quantity in question is computed as the limit, $n\to\infty$, of the corresponding quantity for n successive random variables of the process. Corresponding evaluations are obtained in the continuous-parameter case.

J. L. Doob (Urbana, Ill.).

Blanc-Lapierre, A. Considérations sur la théorie de la transmission de l'information et sur son application à certains domaines de la Physique. Ann. Inst. H. Poincaré **13** (1953), 245–296 (1954).

This paper is in three parts. The first two parts are expositions of Shannon's theory of the transmission of information and certain results of the author [C. R. Acad. Sci. Paris **234**, 1733–1735 (1952); MR **13**, 958] and others concerning properties of random functions whose spectral distribution functions vary only in a bounded interval. In the third part (22 pp.) these results are applied to optics viewed as the study of transmission of information from an object to its image. *R. A. Leibler* (Washington, D. C.).

Kullback, S. Certain inequalities in information theory and the Cramér-Rao inequality. Ann. Math. Statist. **25**, 745–751 (1954).

The main point is the derivation and exploitation of an inequality for a pair of probability distributions on the real

line, which has the inequality referred to in the title as a limiting case. Incidentally, the name used in the title is now known to be misleading [see, e.g., Savage, Foundations of statistics, Wiley, New York, 1954, p. 238; MR 16, 147].

L. J. Savage (Chicago, Ill.).

Landau, H. G. The distribution of completion times for random communication in a task-oriented group. Bull. Math. Biophys. 16, 187–201 (1954).

Given n individuals each having initially a distinct item of information. Let these individuals be connected by some restricted communication net. The task of these individuals is to transmit their items of information by successive messages so that all n individuals acquire all n items of information. A message from A to B always consists of all the items of information that A has so far acquired. All individuals send messages simultaneously at sending times $t_0, t_1, \cdots$ considered to be at unit intervals apart. At each sending time each individual can send a limited number of (identical) messages to recipients chosen in some way within the possibilities afforded by the given communication net. The problem here is to determine the distribution of completion times T of the given task. The following two cases are considered. (1) Each individual can send only one message at a time to a recipient chosen at random from among his possible recipients. Here the solution is formulated in terms of a Markov chain, and the expected value $E(T)$ is tabulated for some simple nets. Generalizations are indicated. (2) Each individual can send at most r messages at a time. Here it is shown that the minimum possible value τ of T is given by the relation

$$\log_{r+1} n \leqq \tau < 1 + \log_{r+1} n,$$

and nets are specified by which τ may be realized.

C. C. Torrance (Monterey, Calif.).

Pollaczek, Félix. Développement de la théorie stochastique des lignes téléphoniques pour un état initial quelconque. C. R. Acad. Sci. Paris 239, 1764–1766 (1954).

In previous work [Ann. Inst. H. Poincaré 11, 135–173 (1949); C. R. Acad. Sci. Paris 236, 578–580 (1953); MR 11, 672; 14, 772] the author has reduced the telephone problem for a simple trunk group with s trunks (the many-server problem of queuing theory) with independent inter-arrival and service times, each with the same (arbitrary) distribution, to the solution of s simultaneous integral equations. Both delay and non-delay systems have been considered and waiting-time distributions and congestion probabilities determined, conditional on an initial state of no trunks busy. In the present paper the modifications necessary to accommodate an arbitrary initial state are given. The initial state is specified by a set of s non-negative numbers t_1 to t_s, the times necessary to complete service on the calls initially present. It turns out that the generating functions for an arbitrary initial state may be expressed in terms of similar functions for the null initial state (the expression is given explicitly but with all accompanying definitions is too long to quote).

J. Riordan (New York, N. Y.).

Blatt, John M. Integral equations for cascade showers. Phys. Rev. (2) 96, 1644–1650 (1954).

Chartres, B. A., and Messel, H. New formulation of a general three-dimensional cascade theory. Phys. Rev. (2) 96, 1651–1654 (1954).

***Marschak, Jacob. Probability in the social sciences.** Mathematical thinking in the social sciences, pp. 166–215, 423–427. The Free Press, Glencoe, Ill., 1954. $10.00.

A lucid expository essay, with section headings: I. Probabilities and the norms of behaviour; II. Probabilities and descriptive social science; III. Probability and policy. We give some examples of points made. In Section I the "norms" of T. Bayes, D. Bernoulli and F. P. Ramsey are described. The Bayes norm defines (subjective) probabilities in terms of decisions and expected monetary gains. The Bernoulli norm defines utilities in terms of given probabilities and decisions. In the Ramsey norm the decisions reveal information concerning the probabilities and the utilities.

In Section II it is questioned whether social scientists are right in drawing on analogies with the kinetic theory of gases.

The following statements are made in Section III. "A good decision-maker dresses up a payoff matrix (though not necessarily with pen on paper). I suspect this skill is not entirely inborn and can be acquired by training: a fruitful field for applied psychology."

I. J. Good.

***Anderson, T. W. Probability models for analyzing time changes in attitudes.** Mathematical thinking in the social sciences, pp. 17–66, 418–419. The Free Press, Glencoe, Ill., 1954. $10.00.

In this paper the author proposes a simple probability model to serve as a basis for analysing changes in attitudes in time. The model is supposed to describe in a certain sense how individuals modify their opinions as time passes. If there are m possible attitudes and if time is measured discretely, the opinion held by a given individual at time t is identified with the state of time t of a system governed by a stationary Markov chain having m possible states. The opinion-histories of the individuals of a group are supposed to be independent realisations of the same Markov chain. If now the ith component of the vector $n(t)$ is the number of people in the group who hold opinion i at time t, then $n(0), n(1), n(2), \cdots$ will be a realisation of a more complicated Markov chain whose structure can be inferred from the transition-matrix for the simpler chain (the one describing the changes in the opinion of a single individual) and the multinomial distribution. The author sketches a number of theoretical properties of the model and then turns to the analysis of an actual example. This consists of the results of interviewing a certain group of 600 people in each of six successive months (the data were collected by the Bureau of Applied Social Research). Questions of estimation, testing and prediction are considered in some detail, and the paper concludes with some notes on rather more general models.

D. G. Kendall (Oxford).

Yntema, L. Einiges zur Wahrscheinlichkeitsansteckung. Verzekerings-Arch. Actuarieel Bijvoegsel 31, 86*–91* (1954).

A Pólya process is a discrete stochastic birth process, the probability of a birth occurring between times t and $t+h$ being $\lambda_n(t)h+o(h)$, where $\lambda_n(t)=(1+an)/(1+at)$ and n births have already occurred [cf., e.g., Feller, Probability theory, v. I, Wiley, New York, 1950; MR 12, 424]. The probability of n births occurring prior to time t may then be written as a compound Poisson process or as a term in the expansion of a binomial with a negative index. The author generalizes $\lambda_n(t)$ to $(1+a_n n)/(1+at)$ and shows that while the negative binomial form still applies, the compound Poisson form is no longer valid.

H. L. Seal.

Mathematical Statistics

Geiringer, Hilda. **On the statistical investigation of transcendental numbers.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 310–322. Academic Press Inc., New York, 1954. $9.00.

About three years ago the first 2000 digits of the transcendentals e and π were calculated on the ENIAC. The author has made a set of statistical evaluations of these results. The paper consists broadly of two parts: the first is concerned with certain general statistical properties as they relate to the problem at hand and the second to some numerical results. The author indicates the present paper is an introductory one and hopes to take up this question again later. *H. H. Goldstine* (Princeton, N. J.).

Kimball, A. W. Short-cut formulas for the exact partition of χ^2 in contingency tables. Biometrics 10, 452–458 (1954).

Abdel-Aty, S. H. Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution. Biometrika 41, 538–540 (1954).

The author determines the first 5 cumulants of the non-central χ^2 distribution with f degrees of freedom with parameter λ after making the Wilson-Hilferty cube-root transformation. He determines two useful approximations to the non-central χ^2 distribution by use of the Cornish-Fisher expansion and illustrates the accuracy of each by calculation of the upper and lower 5% points for $f=2, 4, 7$ and $\lambda=1, 4, 16, 25$ for each value of f. In addition the probability integral is calculated for various values of f, λ, and χ'^2 where χ'^2 denotes the random variable in the non-central χ^2 distribution. *L. A. Aroian* (Culver City, Calif.).

van Rooijen, J. P. On means and the law of errors. Verzekerings-Arch. Actuarieel Bijvoegsel 31, 77*–85* (1954).

The author considers general mean-value functions $f(a_1, a_2, \cdots, a_n)$, that is, functions which are (i) symmetric in all variables, (ii) homogeneous of first degree, and (iii) satisfy $f(a, a, \cdots, a)=a$. A number of such functions are listed and a "general law of errors" is derived from the assumption that a given mean-value function is the most probable value. *E. Lukacs* (Washington, D. C.).

Kuiper, N. H. Note on the fitting of a function to a large number of observations. Statistica, den Haag 8, 1–6 (1954). (Dutch. English summary)

Elementary discussion on the choice of sets of functions for curve fitting. The author also recommends, with a numerical example, a least-squares fit to observed values of y_k in which parameters α, β, γ are calculated by minimising

$$\sum_k \{\alpha\varphi(x_k)+\beta+\gamma x_k-y_k\}^2,$$

where $\varphi(x_k)$ is an empirical function or a set of numerical values only. *M. E. Wise* (London).

Tiago de Oliveira, J. Composite distributions. Their application to ecology. Ciência 4, nos. 9–10, 81–87 (1954). (Portuguese)

Broadbent, S. R. The quotient of a rectangular or triangular and a general variate. Biometrika 41, 330–337 (1954).

The author obtains the distribution function of the quotient y/x, where y is distributed rectangularly in $(1-\alpha, 1+\alpha)$ and x is a variable independent of y with mean 1 and variance β^2 under suitable restrictions on the distribution function of x. The results are generalized to the case where $y=\sum y_i$, $i=1, 2, \cdots, n$, if each y_i is independent and distributed in the same rectangular distribution. The results are used to determine the 1%, 5%, 95% and 99% points if y is rectangular with range $(1-\alpha, 1+\alpha)$ and x is normal with mean 1 and variance β^2; and for the same percentage points of y/x if y is triangular with range $(1-2\alpha, 1+2\alpha)$ and x is normal with mean 1 and variance β^2. In the first case $100\alpha=0(2)10$, $100\beta=0(1)5$, and in the second case $100\alpha=0(1)5$, $100\beta=0(1)5$. An interesting example illustrates the theory. *L. A. Aroian* (Culver City, Calif.).

Cox, D. R. The mean and coefficient of variation of range in small samples from non-normal populations. Biometrika 41, 469–481 (1954).

The author is interested in the mean range and the coefficient of variation of the range in small samples of 2, 3, 4, and 5, from non-normal populations. Different types of populations covering a wide range of values of β and β_2, the usual measures of skewness and kurtosis, are considered: symmetrical and unsymmetrical mixed normal distributions, the normal distribution, the rectangular distribution, exponential type distributions, the Pearson system, and Shone's numerical results for 5 populations of discrete values. Based on these results he provides a table for the normalized mean range and the coefficient of variation of the range to 3 decimals for sample sizes of 2, 3, 4 and 5, for $\beta_2=1.0(.2)2.0(.5)5.0(1.0)9.0$; β_1 is not a determining factor. Comparisons are made between the distribution function of the range and the ratio of two ranges from the exponential e^{-x} and the unit normal population. The theory is applied to the point estimation of dispersion by use of the range, the control chart for the range, the comparison of several ranges by use of the range ratio test and the use of the range in the t-test for non-normal as well as normal distributions. Many short tables give numerical illustrations. The author concludes that if β_2 is known, it is possible to make a rough correction for non-normality in methods that use ranges of small samples, and that on the whole these methods are less affected by non-normality than corresponding methods using variances of small samples. *L. A. Aroian.*

Birnbaum, Z. W., and Rubin, H. On distribution-free statistics. Ann. Math. Statist. 25, 593–598 (1954).

Let $X_1, X_2, \cdots, X_n$ be observations on a one-dimensional random variable X which has the continuous c.d.f. $F(x)$. It was observed in a paper by Birnbaum [same Ann. 24, 1–8 (1953); MR 14, 666] that all distribution-free statistics considered in the literature can be written in the form $\Phi[F(X_1), \cdots, F(X_n)]$, where $\Phi(U_1, \cdots, U_n)$ is a measurable symmetric function defined on the unit cube

$$\{U: 0\leq U_i\leq 1, i=1, \cdots, n\}.$$

Statistics of this form are said to be of structure (d). It is first shown that if a statistic has structure (d) then it is distribution free in the class Ω_2 of all continuous c.d.f.'s. Next it is shown by means of a counter-example that not every distribution-free statistic in Ω_2 is of structure (d). The main object of the paper is to show under what restrictions a distribution-free symmetric statistic has structure (d). This is accomplished by introducing the notion of a strongly distribution-free statistic and by working in the more restricted class Ω^*, consisting of all strictly increasing continuous c.d.f.'s. *B. Epstein* (Detroit, Mich.).

Kendall, M. G., and Sundrum, R. M. Distribution-free methods and order properties. Rev. Inst. Internat. Statist. 21, 124–134 (1953).

The aim of this paper is summarized best, perhaps, by the following introductory statement of the authors: "In the last ten years there has been a great deal of attention given to the types of statistical inference which are independent of parent distributions, particular parent parameters, or other unknown features of a statistical situation which are 'nuisances' in the sense that our ignorance about them impairs the accuracy of the inferences which we wish to make. There has grown up a rather confused terminology, which is regrettable, and an accompanying confusion of thought, which is more than regrettable. In this paper, we attempt a survey of the subject, with the object of clarifying the statistical issues involved and of putting the numerous tests of a 'non-parametric' of 'distribution-free' kind into some sort of framework." In attempting to define precisely what one means by such commonly (and ambiguously) used terms as "parametric", "non-parametric", and "distribution-free", the authors present a number of ideas well worth considering by all statisticians. One interesting assertion made by the authors is the following: "Although it is dangerous to be dogmatic on such points, we are of the opinion that only order properties provide distribution-free tests in the strict sense." *B. Epstein* (Detroit, Mich.).

Kendall, M. G. Two problems in sets of measurements. Biometrika 41, 560–564 (1954).

Two problems called the Angel and the Demon problem are treated. The Angel problem is as follows: "r members are drawn at random from a normal population with unit variance. A benevolent angel tells us which is nearest to the true mean, and the others are rejected. What is the variance of the retained member?" The Demon problem is as follows: "Given a (small) sample of r values from a normal population, what is the probability that their mean lies between the nth and $(n-1)$th in order of magnitude?" Previous writers solved the Angel problem for the cases $n=2$ and 3. The author gives exact results for $n=4$ and 5, a good approximation for moderately large n and an asymptotic result for large n. The Edgeworth form of the Gram-Charlier expansion is used to give explicit values of the probabilities in the Demon problem for the case $n=4(1)10$.

B. Epstein (Detroit, Mich.).

Gnedenko, B. Kriterien für die Unveränderlichkeit der Wahrscheinlichkeitsverteilung von zwei unabhängigen Stichprobenreihen. Math. Nachr. 12, 29–66 (1954). (Russian. German summary)

This is an exposition of results due to the author and his pupils, V. S. Korolyuk, V. S. Mihalevič, and E. L. Rvačeva, which have appeared since 1951 in various papers [see MR 13, 570, 760; 14, 60, 297; 15, 544, 635, 808] on the distribution of the deviation between two empiric distribution functions. *J. Wolfowitz* (Ithaca, N. Y.).

Moore, P. G. A note on truncated Poisson distributions. Biometrics 10, 402–406 (1954).

Generalizing a procedure proposed in his earlier paper [Biometrika 39, 247–251 (1952); MR 14, 391], the author suggests simple estimates for the parameter λ of a Poisson distribution if (1) values $\geqq s$ can not be observed, or (2) values $\leqq k$ are not observed, or (3) values $\geqq s$ and $\leqq k$ are not observed, or (4) values between v and w are not observed. A numerical illustration and a table of asymptotic values of the variance of the estimate for case (2) are presented. No theoretical derivations are given.

Z. W. Birnbaum (Seattle, Wash.).

Hoel, Paul G. A test for Markoff chains. Biometrika 41, 430–433 (1954).

The author derives the asymptotic distribution of the likelihood ratio criterion for testing the hypothesis that an r-dependent Markoff chain with finitely many states is only $(r-1)$-dependent, using methods like Bartlett's [Proc. Cambridge Philos. Soc. 47, 86–95 (1951); MR 12, 512].

J. Kiefer (Ithaca, N. Y.).

Des Raj. On estimating the parameters of binormal populations from linearly truncated samples. Ganita 4, 147–154 (1953).

The object of this paper is to estimate the parameters of the complete bivariate normal population from linearly truncated random samples from it, when the number of unmeasured observations is unknown or not. It is shown that the method of maximum likelihood and the method of moments give identical results. The solution of the estimating equations is made to depend on the functions ψ_1 and ψ_2, extensively tabulated by the author [Ganita 3, 41–57 (1952); MR 14, 569]. Asymptotic variances and covariances of the estimates are obtained. Numerical examples are given to illustrate the application of these results to practical problems. (From the author's introduction.)

Z. W. Birnbaum (Seattle, Wash.).

Page, E. S. An improvement to Wald's approximation for some properties of sequential tests. J. Roy. Statist. Soc. Ser. B. 16, 136–139 (1954).

The author gives an improvement over Wald's approximate formulae for finding the OC and ASN curves associated with a sequential test. *B. Epstein* (Detroit, Mich.).

Leimbacher, Werner R. On some classes of sequential procedures for obtaining confidence intervals of given length. Univ. California Publ. Statist. 2, 1–21 (1953).

Wald [Sequential analysis, Wiley, New York, 1947, pp. 145–150, 155; MR 8, 593] gave a method for obtaining sequential confidence intervals of specified confidence level and length. The author considers three more general methods and then shows, under certain regularity conditions, that any optimum procedure obtainable by these methods is also obtainable by Wald's, and is nonsequential. A broader class of procedures which also contains (in an example) an optimum sequential procedure (Wald's rectangular case as generalized by Blyth [Ann. Math. Statist. 22, 22–42 (1951); MR 12, 622]) is therefore introduced. The proof of optimality of procedures constructed by the author's methods (and which are optimum) is not contained in the methods themselves but is known from results like that cited in the last reference. *J. Kiefer* (Ithaca, N. Y.).

Walter, Edward. Über die Ausnutzung der Irrtumswahrscheinlichkeit. Mitteilungsblatt Math. Statist. 6, 170–179 (1954).

In the testing of statistical hypotheses, particularly for non-parametric tests and tests involving a discrete variate, it is ordinarily not possible to obtain a critical region of exact size α, but rather one which is equal to or less than α. Let y_0 be such that if $y<y_0$, H_0, the hypothesis under test, is accepted and if $y>y_0$, H_0 is rejected. If $y=y_0$, it is assumed that a second random variable y' may be defined such that

if $y' < y_0'$, H_0 is accepted and if $y' > y_0'$, H_0 is rejected. This second variable is so chosen as to maximize the power of the test. If $y' = y_0'$, a third random variable y'' is defined, etc. In this way it is possible to enlarge the critical region which originally was less than α to one which is much closer to α. This idea is applied to R. A. Fisher's sign test, under the assumption that the cumulative distribution is continuous and symmetrical, when it is desired to test the hypothesis that the population mean μ is zero. The second random variable y' is the rank of the absolute value of the largest negative observation and y'' is the rank of the absolute value of the second largest negative observation, etc. In Table one the author gives the critical values of the modified test for both one-sided and two-sided tests when n, the sample size, equals 5(1)25 for $\alpha = .05$; $n = 7(1)25$, $\alpha = .01$; and $n = 10(1)25$ for $\alpha = .001$. The power function is evaluated by experimental sampling using a normal distribution with $\mu = .6$, 1.0 and 1.4 for $\alpha = .05$ and $\alpha = .01$ and compared with the original sign test (to which it is vastly superior) and to the most efficient test, Student's t test, $n = 10$. In a third table the author gives the size of the critical region for $\alpha = .05$, $n = 5(1)30$ and shows that his modified test is monotonically increasing and for $n \geqq 10$ is effectively equal to .05. An example illustrates the method.

L. A. Aroian (Culver City, Calif.).

***De Munter, Paul. Consistance et impartialité des tests non-paramétriques.** Thèse, Université libre de Bruxelles, 1954. 96 pp. (unpaged)

This thesis is a systematic study of unbiasedness and consistency properties of many common nonparametric tests. Most of the considerations are for the 2-sample and r-sample problems. It would take too long to detail here which of the unbiasedness and consistency results are or are not new. The author includes consideration of some new tests, e.g., an r-sample generalization of Whitney's test. Other nonparametric problems are considered in a last chapter which includes a generalization to the curvilinear case of the regression test of Brown and Mood. *J. Kiefer.*

Klerk-Grobben, Gerda, and Prins, H. J. A test for comparing two small unknown probabilities, using samples of equal size, and its power. Statistica, den Haag **8**, 7–20 (1954). (Dutch. English summary)

The tests are actually for differences between observed Poisson means n_1 and n_2 and their use is suggested in binomial samples of N_1 and N_2, both about equal to N, with population means μ_1 and $\mu_2 < 0.1N$. Under the hypothesis $\mu_1 = \mu_2 = \frac{1}{2}m = \frac{1}{2}(n_1 + n_2)$, levels of significance n_α are given for levels α as near as possible (since the n_α are integers) to 0.005, 0.01, 0.025, and 0.05, and $m = 1(1)100$. The corresponding power functions are derived and plotted for $\alpha = 0.025$ and 0.05 and $k = \mu_2/\mu_1 = \frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, and $\frac{2}{3}$; to do this the authors follow Przyborowsky and Wilenski [Biometrika 31, 313–323 (1940); MR 1, 346] except that the latter chose integral n_α to give significances at least as high as α: the true α values are compared in a few cases. [The treatment of a closely related problem by D. R. Cox [ibid. 40, 354–360 (1953); MR 15, 332] should also be noted.]

M. E. Wise (London).

***Elfving, G. Convex sets in statistics.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 34–39 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

In many theories in statistics and mathematical economics, one deals with the maximization of functions on a convex set. As an example, the author gives an expository report on some recent work in design of experiments [see G. Elfving, Ann. Math. Statist. 23, 255–262 (1952); MR 13, 963; and H. Chernoff, ibid. 24, 586–602 (1953); MR 15, 452]. The main result states that if one is interested in s parameters out of k involved in a set of available experiments, there is an optimal design which involves repeating at most $k + (k-1) + \cdots + (k-s+1)$ of these experiments.

H. Chernoff (Stanford, Calif.).

Konijn, H. S. On certain classes of statistical decision procedures. Ann. Math. Statist. **24**, 440–448 (1953); corrections **25**, 170–171 (1954).

In a statistical decision problem, let F denote the class of distributions f and D the class of decision functions d. Let $W(f, d)$, $c(f, d)$ denote the (random) loss from terminal decisions and cost of experimentation, respectively, when using d against f. In his study of statistical decision functions, Wald [Statistical decision functions, Wiley, New York, 1950; MR 12, 193] formulated certain detailed hypotheses H on f, D, W, c, under which most of his results were obtained. The author investigates whether, assuming H to hold, it continues to hold when D is replaced by certain subclasses of interest. The answer is affirmative for, among others, the classes defined by $EW(f, d) \leqq y_0$ for all f, $Ec(f, d) \leqq z_0$ for all f, $\Pr\{c(f, d) > z\} \leqq \beta$ for all f and, except for a countable set of y's, $\Pr\{W(f, d) > y\} \leqq \alpha$ for all f.

D. Blackwell (Berkeley, Calif.).

Turner, C. H. M. On the concept of an instantaneous power spectrum, and its relationship to the autocorrelation function. J. Appl. Phys. **25**, 1347–1351 (1954).

A definition of the instantaneous power spectrum of a signal $G(t)$ [C. H. Page, J. Appl. Phys. 23, 103–106 (1952)] is considered and three questions concerning its uniqueness, its value as given by observers at two different positions in time, and its relation to a "running autocorrelation function $A(T, X)$" are answered. $A(T, X) = \int_{-\infty}^{\infty} G_T(t) G_T(t+x)\,dt$, where $G_T(t) = G(t)$ for $t \leqq T$ and is zero elsewhere.

R. A. Leibler (Washington, D. C.).

Have

***Masuyama, Motosaburo. Graphical method of statistical inference. Seminar note.** Maruzen Company, Ltd., Tokyo, 1954. ii+83 pp. (1 plate). $2.50; 400 yen.

The author states the aim of this seminar note is to make graphical methods available for medical, biological and engineering students. The author confines himself to graphical methods using double square root paper; the basic exposition of such methods is due to Mosteller and Tukey [J. Amer. Statist. Assoc. 44, 174–212 (1949)]. A chapter is devoted to some extensions of the theory of this technique, based on various normal approximations to common statistics, which are known though they have not been exploited in this direction heretofore. The actual applications are given in hints and answers to three chapters of exercises. These can be comprehensible only to those with a fairly well developed background in statistics. Consequently the book can hardly be recommended to the beginning student or even one who has been exposed to an introductory course in statistics. Those familiar with statistical theory, who are interested in learning graphical methods might perhaps find this a useful monograph. *D. G. Chapman* (Oxford).

Vincze, István. On mathematical statistical methods of quality control of mass production. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 429–441; discussion 442–444 (1954). (Hungarian)

This is a lecture on statistical quality control given before the Hungarian Academy of Sciences. A considerable part of the paper describes the work of the Institute of Applied Mathematics in this field. An illustrative example of some interest is given: the author considers the control procedure for a manufacturing process under the assumption that the number of defectives follows a Pólya distribution.

E. Lukacs (Washington, D. C.).

Oderfeld, J. On the automatization of calculations in statistical quality control. Zastos. Mat. 1, 188–196 (1954). (Polish. Russian and English summaries)

Drobot, S., and Warmus, M. Dimensional analysis in sampling inspection of merchandise. Zastos. Mat. 2, 1–33 (1954). (Polish. Russian and English summaries)

Polish version of a paper in Rozprawy Mat. 5 (1954); MR 16, 55.

Mathematical Economics

Berge, Claude. Sur une théorie ensembliste des jeux alternatifs. J. Math. Pures Appl. (9) 32, 129–184 (1953).

The author considers a two-person game, defined by a set X, a point $x_0 \varepsilon X$, and a function Γ, mapping points of X into subsets of X, and played as follows: at move i $(=1, 2, \cdots)$ player I (for i odd) or II (for i even) chooses a point $x_i \varepsilon \Gamma(x_{i-1})$; play stops when a point x_i is reached for which $\Gamma(x_i)$ is empty (this may never happen). Thus the games considered are games of perfect information, with a possibly infinite set of choices at each move, and with the possibility of indefinite continuation.

The author is concerned with describing properties of the game in terms of the function Γ; for instance, if S is a specified set of terminal positions (positions s with $\Gamma(s)=\text{null}$) and T is the set of positions from which player I can force eventual termination in S, a formula is given for T in terms of S, Γ and related quantities. It is shown that, for any set S of terminal positions, either player I can force eventual termination in S or player II can prevent termination in S.

Let $A=\|a_{ij}\|$, $i, j=1, 2, \cdots$, be the matrix of a zero-sum two-person game with a countable number of strategies for each player. If A, considered as an operator in Hilbert space, is totally continuous, the game with matrix A is shown to have a value. *D. Blackwell* (Berkeley, Calif.).

Berge, Claude. Le problème du gain dans la théorie généralisée des jeux sans informations. Bull. Soc. Math. France 81, 1–8 (1953).

In the paper reviewed above, the author has proposed a general formal treatment of zero-sum two-person games in extensive form. The distinctive features of this formulation are: (1) cycles of choices may occur; (2) a play may contain an infinite sequence of choices; and (3) the number of alternatives presented to a player at a move may be infinite. The problem considered in the present work is that of the relation of $h(t)$, the best assurance of a player in a situation t, with the information D, the payoff f, and the transform G that defines the game. A criterion is developed for the game to be perfect (i. e., h independent of D), with (2) above replaced by the customary stop rule. *H. W. Kuhn.*

*Simon, Herbert A. **Some strategic considerations in the construction of social science models.** Mathematical thinking in the social sciences, pp. 388–415, 435–438. The Free Press, Glencoe, Ill., 1954. $10.00.

It is shown by examples how the essential elements of a wide range of postulates about human behavior, individual and social, can be translated into mathematical models, and it is indicated how such models may be used effectively. Three canons of strategy are stated as guides in the formulation of such models. The models exhibited range from simple processes of maximization to complex processes of adaptation. *C. C. Torrance* (Monterey, Calif.).

Isard, Walter. A general location principle of an optimum space-economy. Econometrica 20, 406–430 (1952).

Suggests that many of the special problems of locational economics have a common variational basis, namely that locations and boundaries between market areas are to be chosen to maximize an integral which represents total rent or surplus for society. The work is mainly formal.

R. Solow (Cambridge, Mass.).

Have Eng.

***Wold, Herman O. Demand analysis, part I.** Trabalhos do seminário de econometria dirigido pelo Prof. H. O. Wold [Works of the seminar on econometrics led by Prof. H. O. Wold], pp. 7–116. Publicações do Centro de Estudos Económicos, Lisbon, 1953. (Portuguese)

Translation into Portuguese of Part I of the author's "Demand analysis" [Gebers, Stockholm, 1952; Wiley, New York, 1953; MR 16, 274].

Debreu, Gerard. Valuation equilibrium and Pareto optimum. Proc. Nat. Acad. Sci. U. S. A. 40, 588–592 (1954).

For an economic system with given technological and resource limitations, individual needs and tastes, a valuation equilibrium with respect to a set of prices is a state where no consumer can make himself better off without spending more, and no producer can make a larger profit; a Pareto optimum is a state where no consumer can be made better off without making another consumer worse off. The author gives the conditions under which a valuation equilibrium is a Pareto equilibrium (the convexity of the consumption sets and the convexity of preferences) and the conditions under which a Pareto equilibrium is a valuation equilibrium (continuity of preferences, the convexity of the aggregate production-set, and a linear space L in which an economic system is considered is finite-dimensional and/or has an interior point). Thus the author extends the results of Arrow [Proc. 2nd Berkeley Symposium on Math. Statist. and Probability, 1950, Univ. of Calif. Press, 1951; MR 13, 482] and Debreu [Econometrica 19, 273–292 (1951)] so as to make the results applicable to the problem of infinite time horizon.

S. Ichimura (Osaka).

Marschak, J., and Mickey, M. R. Optimal weapon systems. Naval Res. Logist. Quart. 1, 116–140 (1954).

Expected military utility $E[u]$ is expressed in terms of damage probability and maximized subject to cost. Some simple numerical examples are worked out. The basic ideas are analyzed and extended in some detail.

C. C. Torrance (Monterey, Calif.).

Solomon, Morris J. Optimum operation of a complex activity under conditions of uncertainty. J. Operations Res. Soc. Amer. 2, 419–432 (1954).

The activity discussed is the flying of planes at a base, the subactivities considered being the supply of spare parts.

Reorder points for parts are determined that yield minimum routine supply cost and provide for a specified probability against a plane being grounded for lack of one or more spare parts. A procedure is developed for determining the optimum such reorder points on the basis of overall costs resulting from grounded planes. *C. C. Torrance.*

***Operations research for management.** Edited by Joseph F. McCloskey and Florence N. Trefethen. The Johns Hopkins Press, Baltimore, Md., 1954. xxiv+409 pp. $7.50.

This is a collection of lectures given in a seminar on operations research held at The Johns Hopkins University in the spring of 1952. The following are of possible mathematical interest: Queueing theory (pp. 134–148) by B. O. Marshall, Jr.; Information theory (pp. 149–167) by D. Slepian; Symbolic logic in operations research (pp. 187–202) by W. E. Cushen; The use of computing machines in operations research (pp. 203–216) by J. O. Harrison, Jr.; Linear programming and operations research (pp. 217–237) by J. O. Harrison, Jr.; and Game theory (pp. 238–253) by D. H. Blackwell.

***Proceedings of the symposium on operations research in business and industry, April, 1954.** Midwest Research Institute, Kansas City, Mo., 1954. iv+185 pp. $5.00.

These proceedings contain twelve papers on the technique and application of operations research. Of possible interest to mathematicians are: Computational experience in solving linear programming problems (pp. 92–104) by W. Orchard-Hays; Problems of traffic and transportation (pp. 105–113) by W. Prager.

***Proceedings of operations research conference by Society for Advancement of Management, New York, 1954.** Society for Advancement of Management, New York, 1954. 356 pp. (mimeographed). $15.00.

These proceedings consist of a number of papers on the use, techniques and applications of operations research. They are nonmathematical in character.

TOPOLOGY

Garcia Tranque, Tomás. The type in cubic graphs. Gaceta Mat. (1) **5**, 11–23 (1953). (Spanish)

Se bornant aux graphes dont chaque point est au moins de degré 2, l'auteur appelle type d'un point x du graphe la suite des nombres rangés par ordre croissant obtenus en considérant l'ensemble des couples d'arètes ayant x pour extrémité et en associant à chacun de ces couples la longueur du plus petit circuit du graphe qu'elle détermine. Utilisant alors la remarque évidente que des points de type différents ne peuvent faire partie d'une même classe de transitivité du groupe d'automorphisme, il donne explicitement un graphe cubique de $6h$ sommets ayant pour groupe d'automorphismes le groupe cyclique d'ordre h.

A signaler: 1) de nombreuses erreurs typographiques, 2) dans la figure 3, page 14, la réconstitution des capitales absentes qui doivent désigner les 12 points offre au lecteur une distraction genre "mots croisés". *J. Riguet* (Paris).

Tihomirova, E. S. Infinitesimal classification of surfaces of second degree. Uspehi Mat. Nauk (N.S.) **9**, no. 1(59), 121–123 (1954). (Russian)

The "infinitesimal" (in the sense of proximity spaces) classification of quadric surfaces (in the real space) differs from the well-known topological one as follows: the elliptic cylinder and the hyperboloid of one-sheet belong to different equimorphism classes (a one-to-one transformation f is an equimorphism if both f and f^{-1} are uniformly continuous); the hyperbolic paraboloid, the parabolic cylinder and the plane are equimorphic whereas the elliptic paraboloid belongs to another equimorphism class. *M. Katětov.*

Valle Flores, Enrique. Observation on a theorem of D. Ellis. Bol. Soc. Mat. Mexicana **10**, nos. 3–4, 31–32 (1953). (Spanish)

The Baire (metric) space over a set S has as points infinite sequences of elements of S with $d(a,a)=0$ and $d(a,b)=1/k$, where $a=(a_1, a_2, \cdots)$, $b=(b_1, b_2, \cdots)$, $a_i, b_i \in S$, $i=1, 2, \cdots$, and k is the first index such that $a_k \neq b_k$. Ellis showed that each finite or countable group G is isomorphic to a subgroup of the group of motions of $\mathfrak{B}(G)$, the Baire space over G [Math. Mag. **26**, 183–184 (1953); MR **14**, 843]. This note points out that no restriction of the cardinality of G is required. *L. M. Blumenthal.*

Tits, J. Sur un article précédent: "Etude de certains espaces métriques". Bull. Soc. Math. Belgique **6** (1953), 126–127 (1954).

See same Bull. **5**, 44–52 (1952); MR **15**, 334.

Morita, Kiiti. Normal families and dimension theory for metric spaces. Math. Ann. **128**, 350–362 (1954).

M. Katětov has shown that a large part of dimension theory, originally proved only for separable metric spaces, can be extended to arbitrary metric spaces [Dokl. Akad. Nauk SSSR (N.S.) **79**, 189–191 (1951); Čehoslovack. Mat. Ž. **2**(77), 333–368 (1953); MR **15**, 145, 815]. The present paper derives substantially the same results by different methods, which depend on generalizing Hurewicz's "Normalbereiche" [Math. Ann. **96**, 736–764 (1927)]. In what follows, all spaces occurring are to be metrizable. A family $\mathfrak{M}$ of spaces is "normal" if (a) $Y \in \mathfrak{M}$ whenever $Y \subset X \in \mathfrak{M}$, (b) if X has a locally countable closed covering $\{A_\alpha\}$, $A_\alpha \in \mathfrak{M}$, then $X \in \mathfrak{M}$. It is enough that (b) hold both when $\{A_\alpha\}$ is countable and also when $\{A_\alpha\}$ is locally finite. Given a normal family $\mathfrak{M}$, let $\mathfrak{M}'$ be the family of spaces X with the property: every closed subset of X has arbitrarily fine open neighborhoods in X with frontiers in $\mathfrak{M}$. Then $\mathfrak{M}'$ is a normal family, and consists of all spaces of the form $Y \cup Z$, where $Y \in \mathfrak{M}$ and $\dim Z=0$. This gives quick proofs of the principal properties of the dimension-function $\operatorname{Ind} X$ (called ind dim X in the present paper), defined inductively in terms of frontiers of neighborhoods of closed sets. Next the covering dimension $\dim X$ is shown to equal $\operatorname{Ind} X$ for all X; hence the spaces with $\dim X \leqq n$ form a normal family for each n, and the sum, product and decomposition theorems are valid. An imbedding theorem for 0-dimensional spaces (also due to Katětov) is obtained, and the following generalization is given: an S-space (i.e., space with the star-finite property) can be imbedded in the product of the Hilbert cube with countably many discrete spaces. Further results include theorems asserting the existence of coverings and decompositions of X which behave suitably for $\aleph_0$ closed finite-dimensional subsets of X simultaneously. The status of the Menger-Urysohn dimension function ind X (denoted by ind dim* X in the present paper), defined inductively in terms of frontiers of neighbourhoods of points, remains open

in general; but the author extends a result of G. H. Butcher [Duke Math. J. 18, 859–874 (1951); MR 13, 573] by proving that ind $X=\dim X$ if X is the union of countably many closed sets each of which is an S-space. *A. H. Stone.*

Toulmin, G. H. Shuffling ordinals and transfinite dimension. Proc. London Math. Soc. (3) 4, 177–195 (1954).

"Shuffling" is defined as follows: γ shuffles α, β if there are ordered sets A, B, C of order-types α, β, γ, respectively, such that $C=A\cup B$, $A\cap B=0$. If α, β are ordinals, then any γ shuffling α, β is an ordinal; the least (respectively, the sup) of such γ, denoted $\alpha\pm\beta$ (respectively, $\alpha\mp\beta$) is called the lower (upper) sum of α and β.

Properties of $\alpha\pm\beta$ and $\alpha\mp\beta$ are considered. Some results: $\alpha\pm\beta=\beta\pm\alpha$; if $\alpha\leqq\beta$, $\alpha=\alpha_1+\alpha_2$, $\beta=\beta_1+\beta_2$, α_1, β_1 are limit ordinals, $\alpha_2<\omega$, $\beta_2<\omega$, then $\alpha\pm\beta=\beta$ if $\alpha_1<\beta_1$, $\alpha\pm\beta=\beta+\alpha_2$ if $\alpha_1=\beta_1$; $\alpha\mp\beta$ coincides with the natural sum" [cf. G. Hessenberg, Grundbegriffe der Mengenlehre, Vandenhoeck and Ruprecht, Göttingen, 1906, and F. Hausdorff, Mengenlehre, 2. Aufl., de Gruyter, Berlin-Leipzig, 1927]; $\alpha\mp\beta$ is the least function of α, β strictly increasing with each of them. It is shown that the set of ordinals shuffling given α and β is finite.

In the second part of the note, the author gives an inductive definition of the ordinal dimension of an arbitrary topological space: $\dim X\leqq\alpha$ if X has an open base consisting of sets G with $\dim(\bar{G}-G)<\alpha$. The following sum theorems are proved. $1+\dim(A\cup B)\leqq(1+\dim A)\mp(1+\dim B)$; if A, B are closed in $A\cup B$, then

$$\dim(A\cup B)\leqq\max(\dim A, \dim B)\pm(\dim(A\cap B)+1).$$

An example is given of a separable metric space $X=A\cup B$, A, B closed in X, with $\dim X=\omega+1$, $\dim A=\dim B=\omega$. It is proved that the dimension of the topological product (X, Y) does not exceed $\dim X\mp\dim Y$ by more than a finite number (specified in the note); if X is metric and Y is an Euclidean space, then $\dim(X, Y)\leqq\dim X+\dim Y$.

M. Katětov (Prague).

Mardešić, Sibe. Sur les sous-espaces linéaires, singuliers par rapport à un ensemble compact. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 9, 35–39 (1954). (Serbo-Croatian summary)

Let E be a non-empty compact subset of the Euclidean n-dimensional space V^n, $n\geqq1$. A subspace V^m of V^n, $0\leqq m\leqq n$, is said to be singular with respect to E if and only if for every linear variety $L=c+V^m$, $c\in V^n$, parallel to V^m, the relation $L\cap E\neq0$ implies that the linear dimension of $L\cap E$ is equal to m. The author proves that the set of all subspaces V^m of V^n, $0\leqq m\leqq n$, which are singular with respect to a compact subset E of V^n is at most denumerable. For the particular case $n=2$, this theorem has been proven by Sierpiński, [Matematiche, Catania 6, 132–134 (1951); MR 13, 673], who proved at the same time that it holds for $m=1$, $n=3$. S. Straszewicz [Fund. Math. 24, 139–143 (1935)] has remarked that the same arguments can be extended without difficulty to the case $m=1$, n arbitrary. The present theorem can be considered as an extension of the results of Sierpiński and Straszewicz. *D. W. Hall.*

Keldyš, Lyudmila. On the representation of zero-dimensional open mappings as superpositions. Dokl. Akad. Nauk SSSR (N.S.) 98, 719–722 (1954). (Russian)

It is proved that every light open map f of a compact metric space X of dimension n onto a compact metric space Y of dimension $m>n$ can be expressed as the composition of a finite number of maps, each of which either (a) does not raise dimension, or (b) has the inverse of any point consisting of at most two points. In previous work [same Dokl. (N.S.) 68, 989–992 (1949); MR 11, 381], the author has shown that every light map possessing a certain property (γ) can be so decomposed. Here she reduces the case of the light open map to the case treated earlier. *E. E. Floyd.*

Yang, Chung-Tao. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I. Ann. of Math. (2) 60, 262–282 (1954).

The author proves a number of theorems containing ones proved by those whose names are listed in the title. This field started with a conjecture made by Rademacher that there exists a frame on a sphere where a given real-valued function takes the same value. This paper deals with a case where an involutive homeomorphism acts on the sphere. The author establishes a homology theory which is essentially that of the space of orbits of the homeomorphisms. Among many noteworthy generalizations of classical lemmas, it seems that the most powerful one is the Theorem 2, which claims that if f is a map from the space of orbits of an involutive homeomorphism on a finite-dimensional space X into k-dimensional euclidean space R^k, then the dimension of the set of points whose orbit goes to a single point in R_f^k, is larger than $n-k$. *H. Yamabe* (Minneapolis, Minn.).

Zarankiewicz, K. Un théorème sur l'uniformisation des fonctions continues et son application à la démonstration du théorème de F. J. Dyson sur les transformations de la surface sphérique. Bull. Acad. Polon. Sci. Cl. III. 2, 117–120 (1954).

The author proves a theorem which is a generalization of a theorem of Dyson [Ann. of Math. (2) 54, 534–536 (1951); MR 13, 450] by making use of a lemma of Kuratowski which shows the existence of a connected component of F or S_n-F if F is an antipodal closed set on the n-sphere S_n. The same theorem was proved by Livesay [Bull. Amer. Math. Soc. 58, 492 (1952)]. Theorems of the paper reviewed above include these theorems. *H. Yamabe.*

Reifenberg, E. R. On the Cartwright-Littlewood fixed point theorem. Ann. of Math. (2) 61, 137–139 (1955).

Let I be a bounded plane continuum possessing only one complementary domain and f an orientation-preserving automorphism of the plane leaving I invariant. The theorem in question asserts that f leaves fixed some point of I. The present short proof follows the scheme of the proof of an early fixed-point theorem of Brouwer [Akad. Wetensch. Amsterdam, Proc. 13, 767–777 (1911)]. A somewhat different proof was recently given by O. H. Hamilton [Canad. J. Math. 6, 522–524 (1954); MR 16, 276]. A contribution by M. H. A. Newman to this proof was acknowledged by Hamilton and should have been mentioned in the review.

P. A. Smith (New York, N. Y.).

Weier, Josef. Sur les classes essentielles des coïncidences de deux représentations. C. R. Acad. Sci. Paris 239, 337–339 (1954).

The author considers two compact topological n-dimensional manifolds P_1 and P_2, where $n\geqq3$, and continuous mappings of P_1 into P_2. A point a of P_1 is a coincidence of two such mappings g, h if $g(a)=h(a)$. Two coincidences a, b belong to the same class if there is a path c from a to b such that the loop $g(c)\cdot h(c^{-1})$ is contractible. The index of

an isolated coincidence a is defined to be the index of the fixed point a of the mapping $f: p \to p+(g(p)-f(p))$, defined for p in a neighborhood of a. It is announced that the number of coincidence classes is finite, and that the number η of essential classes (those for which the index is non-zero) is the minimum number of coincidence points possessed by any pair of mappings g', h' homotopic to g, h. If $P_1=P_2$ and h is the identity map, the notion of coincidence point reduces to that of fixed point, and the announced results to known theorems of Nielsen and Wecken [cf. Wecken, Math. Ann. **118**, 544–577 (1942); MR **5**, 275].

R. H. Fox (Princeton, N. J.).

Weier, Joseph. Sur une propriété des représentations de variétés en variétés. C. R. Acad. Sci. Paris **239**, 1111–1113 (1954).

Let $[f]$ be a class of mappings of a closed oriented m-dimensional manifold P into a closed n-dimensional manifold Q, and suppose $m=n+1$. It is asserted that there are representative maps f^1 and f^2 in this class whose set of coincidences is a finite disjoint collection of simple closed curves $S_1, \cdots, S_p$. To each curve S_j is associated a degree, whose sign depends on the orientation of S_j (and on that of P and Q), which is, crudely speaking, the index of the coincidence of $f^1|T$ and $f^2|T$ for an n-cell T normal to S_j [cf. the preceding review]. Two of these curves S_j and S_k belong to the same class if they are homotopic in P; the curves belonging to a class are to be similarly oriented. A class is essential if the sum of the degrees of its members is not zero. The author announces that the number of essential classes is a homotopy invariant (i.e. does not depend on the representative maps f^1, f^2) and defines this number to be the index of f. This generalizes, in some sense, a theory that is classical for the case $m=n$ and trivial for the cases $m<n$. The author asserts that the cases $m>n+1$ can be treated similarly, and that the conditions of orientability and boundarylessness of P and Q can be circumvented.

R. H. Fox.

Baum, John D. Asymptoticity in topological dynamics. Trans. Amer. Math. Soc. **77**, 506–519 (1954).

The author considers transformation groups whose phase spaces are uniform spaces and whose phase groups are abelian, separable and generated by some compact neighborhood of the identity element. In this setting he proves theorems about various kinds of asymptoticity, these notions being natural generalizations of notions for flows. Typical theorem: Two points are totally asymptotic if and only if they are compactly asymptotic. An existence theorem on asymptotic points is also proved. The author constructs an analogue of the Morse minimal orbit-closure whose phase group consists of the integer points of the plane and analyzes its structure, particularly with reference to its asymptotic properties.

W. H. Gottschalk.

Èl'sgol'c, L. È. Estimation of the number of critical points of a continuous mapping of a manifold onto a circle. Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7**, 34–38 (1954). (Russian)

Let f be a C^2 mapping of a compact manifold M^n onto the circle S^1. A critical point of f is a point where $df=0$; the corresponding image in S^1 is called a critical value. Let p_1 and p_2 be chosen on S^1 in such a way that on one of the two resulting closed arcs there is no critical value of f. If ϕ maps S^1 onto $[0, 1]$, with $\phi(p_1)=0$, $\phi(p_2)=1$, and no other critical points, the Morse theory may be applied to ϕf giving the estimate

$$\sum_{i=0}^{n} m^i(M^n) \geq \sum_{i=0}^{n} p^i(M^n) - \sum_{i=0}^{n-1} m^i(M_1^{n-1}) - \sum_{i=1}^{n} m^i(M_2^{n-1}).$$

Here $M_j^{n-1}=f^{-1}(p_j)$, and $p^i(M)$, $m^i(M)$ are the ith Betti and type numbers, respectively, of the set M. By a closer examination of the increasing and decreasing type numbers of the sets M_j^{n-1} a final estimate is obtained which depends only on the dimensions of various relative cycle groups. In the concluding paragraphs the author estimates the number of geometrically distinct critical points by means of his relative category theory [Mat. Sb. N.S. **8**(50), 455–461 (1940); MR **2**, 325].

L. W. Green.

Oniščik, A. L. On the orientability of analytic homogeneous manifolds. Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 121–130 (1953). (Russian)

Orientation of a manifold is defined in the usual way by means of Jacobians. If the manifold is analytic and homogeneous the 'question of orientability is reduced to the study of the inner automorphisms of [a certain subgroup of the Lie group that is acting on the homogeneous manifold]'. The reviewer cannot find anything new in this paper.

R. H. Fox (Princeton, N. J.).

Gordon, I. I. On coverings of spheres. Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 147–152 (1953). (Russian)

The Hopf theorem: A unicoherent locally connected continuum has no free covering by three closed sets [Fund. Math. **28**, 33–57 (1937)] is proved by a different method for the special case of a polyhedron. The Hopf estimate: $\nu(n) \leq n+1$ of the least number ν for which there is a free covering of S^n by ν closed sets is improved (for $n \geq 7$) to $\nu(n) \leq n+2-[\frac{1}{2}(n+1)]$. Two conjectures of M. A. Krasnosel'skiĭ [Uspehi Mat. Nauk (N.S.) **6**, no. 5(45), 162–165 (1951); MR **13**, 484] are shown to be false.

R. H. Fox.

Ganea, Tudor. Symmetrische Potenzen topologischer Räume. Math. Nachr. **11**, 305–316 (1954).

The author proves several propositions about the nth symmetric power $E(n)$ of a space E as defined by Borsuk and Ulman [Bull. Amer. Math. Soc. **37**, 875–882 (1931)] of which the following are a sample. 1. If E is a connected, locally connected, unicoherent, or simply connected Hausdorff space then $E(n)$ also has these properties. 2. If E is metric, separable, then $\dim E(n)=\dim E^n$, where $E^n=E\times\cdots\times E$ (n factors). 3. If E^n is a Cantor manifold, then $E(n)$ is also. 4. If E is a contractible or locally contractible Hausdorff space, then $E(n)$ is also.

R. Bott.

Eckmann, B. Räume mit Mittelbildungen. Comment. Math. Helv. **28**, 329–340 (1954).

A space R is said to admit an n-mean, if there exists a map $M: R^n \to R$ ($R^n=R\times\cdots\times R$) which is symmetric in the n arguments and whose restriction to the diagonal Δ of R^n is the identity map. The main result of the author is the following theorem. If R admits an n-mean ($n \geq 2$), then $\pi_1(R)$ is abelian and multiplication by n is an automorphism in all the groups $\pi_n(R)$, $H_n(R)$ (integer coefficients).

R. Bott.

Aleksandrov, P. On combinatorial topology of nonclosed sets. Mat. Sb. N.S. **33**(75), 241–260 (1953). (Russian)

The author continues his work on combinatorial topology of arbitrary subsets of n-dimensional spherical space S^n [cf. Mat. Sb. N.S. **21**(63), 161–232 (1947); MR **9**, 456] and

discusses the relation of his theory to that of Sitnikov [Dokl. Akad. Nauk SSSR (N.S.) **81**, 359–362 (1951); MR **13**, 860]. §1: A triangulation is a geometrical complex in S^n locally finite at each of its points; a triangulation is canonical relative to a set $A \subset S^n$ if every principal simplex meets A. Theorem 1 states roughly that the canonical triangulations are cofinal in the directed set of all triangulations containing A, and also cofinal in the directed set of nerves of open coverings of A. §2: Let a homology theory for complexes (and simplicial maps) be given; associate with A the limit group of the homology groups of triangulations containing A (the external homology group) and the limit group of the homology groups of nerves of coverings (internal homology); Theorem 1 leads at once to Theorem 2 (general invariance theorem): internal and external groups are isomorphic. §3 describes the two basic methods of defining homology for an arbitrary set A: (1) by considering the directed set of compact subsets of A; this leads to homology and cohomology "with compact carriers" (note that for cohomology a quite different definition of compact carriers is also in use); the modification in the definition of ~ 0, due to Sitnikov, is described; (2) by considering the directed set of nerves of (star-finite countable) coverings. A proof is given of the author's duality theorem which says that the pth homology group of A, based on method (2) (finite chains) is isomorphic with the $(n-p-1)$th cohomology group "with compact carriers" of the complement B of A. The same method of proof gives an isomorphism between the pth cohomology group of A, based on coverings, and the $(n-p-1)$th homology group "with compact carriers" of B, provided the coefficient group is compact. This is a special case of Sitnikov's duality theorem; his general theorem makes no assumption on the coefficients, but makes use of his own modified definition of homology and cohomology. Sitnikov's theorem is stronger than the author's although it is not a direct strengthening of it. The author's theorem is something like a weak dual of Sitnikov's. The homology group used by the author is not exactly the dual of the cohomology group used by Sitnikov; its character group is only (the completion of) a certain factor group of Sitnikov's cohomology group, with a similar situation for the other pair of groups in the two duality theorems. This comes from the fact that the homology group of B "with compact carriers" contains as (in general, non-trivial) subgroup the annihilator (with respect to the linking coefficient) of the homology group (based on coverings) of A; this is called the subgroup of "non-linking cycles". Other variants of homology groups are discussed, with the help of a theorem on generalized limit groups. §4 describes a new topologically invariant property (e^p) of an arbitrary set A in S^n: every neighborhood λ of A contains a neighborhood λ' such that every (finite polyhedral) p-cycle in λ' is ~ 0 in λ. Invariance is proved by the methods developed earlier, by showing the equivalence of (e^p) with the property (E^p) obtained by replacing neighborhoods by coverings. A modified property (e'^p) (considering relative p-cycles module an arbitrary closed set ϕ) can be used to characterize the dimension of closed sets; whether this extends to arbitrary sets is an open question. §5: Sitnikov has proved, with his definition of homology, the "theorem on obstructions": If $A \subset S^n$ is r-dimensional, then (1) every cycle of dimension $q<n-r-1$, which is ~ 0 in an open set T, is ~ 0 in $T-A$, (2) there exists an open sphere U^n and an $(n-r-1)$-cycle in U^n-A, which is not ~ 0 in U^n-A [ibid. **82**, 845–848; **83**, 31–34 (1952); MR **13**, 860]. The question arises whether one can use cycles with "compact carriers" in this theorem. S. Kaplan has shown this for part (1) [Trans. Amer. Math. Soc. **62**, 248–271 (1947); MR **9**, 456]. Here this is shown to hold for (2), provided A contains a compact r-dimensional subset. At the same time a criterion is obtained for this last property in terms of linking cycles. *H. Samelson* (Ann Arbor, Mich.).

Ehresmann, Charles. Structures locales. Ann. Mat. Pura Appl. (4) **36**, 133–142 (1954).

This paper is identical with one reviewed earlier [Colloque de topologie et géométrie différentielle, Strasbourg 1952, no. 10, Bibliothèque Nat. Univ. Strasbourg, 1953; MR **15**, 731]. *H. Samelson* (Ann Arbor, Mich.).

Munkres, James. The special homotopy addition theorem. Michigan Math. J. **2** (1953–54), 127–131 (1955).

The author gives a proof for the special homotopy addition theorem required to adapt the Eilenberg proof [Ann. of Math. (2) **45**, 407–447 (1944), pp. 439–444; MR **6**, 96] of Hurewicz's isomorphism theorem to the case where the cubical homology theory of the space is used and cubes are taken as anti-images for the homotopy groups. *J. Dugundji* (Los Angeles, Calif.).

Liao, S. D. On the topology of cyclic products of spheres. Trans. Amer. Math. Soc. **77**, 520–551 (1954).

In this paper the author studies the cyclic and symmetric products of spaces, in particular, the cyclic products of spheres. The content of the paper is as follows.

Chapter I. Let Γ be a subgroup of Σ_q, the symmetric group of degree q. Let X be a topological space and X^q the q-fold Cartesian product of X. In a natural fashion Γ can be regarded as a transformation group acting on X^q. The orbit space, denoted by X^Γ, is called the Γ-product of X. For K a complex, the author gives a simplicial decomposition of K^Γ based on a natural decomposition of the Cartesian product K^q. For X a locally compact, paracompact Hausdorff space, the author considers a generalization of Smith's special cohomology theory that he will use later.

For X a connected space, the author proves the existence of an into-isomorphism μ: $H^s(X,G) \to H^s(X^\Gamma,G)$, that satisfies naturality conditions. Then, he introduces the notion of "regular imbedding of X into X^Γ", as follows. Let $\bar{I}$: $X^q \to X$ be the natural map. For fixed $s_0 \in X$, we identify $x \in X$ with $\bar{I}(x, s_0, \cdots, s_0) \in X^\Gamma$. In this form X is topologically imbedded in X^Γ. Let η: $H^s(X^\Gamma,G) \to H^s(X,G)$ be the projection homomorphism obtained from the imbedding. It is shown that $\eta\mu$ is the identity isomorphism of $H^s(X,G)$.

Chapter II. This chapter begins with a generalization to the relative case of some of the author's results on periodic maps acting on a homology sphere [Ann. of Math. (2) **56**, 68–83 (1952); MR **14**, 73]. Let $\vartheta_{np}=p$-fold cyclic product of an n-sphere S_n. The generalizations mentioned above are then used to obtain a complete computation of the cohomology groups of ϑ_{np}, with p prime, $n \geqq 2$, and coefficients Z, the integers, or Z_p, the integers mod p. We have $H^s(\vartheta_{np},Z)=0$ for $0 \leqq s<n$, and $H^n(\vartheta_{np},Z) \approx Z$ generated by $g_n{}^*$. Let $g_n{}^\#$ be the generator of $H^n(\vartheta_{np},Z_p)$ obtained as $g_n{}^*$ mod p. The author gives explicitly the Steenrod cyclic reduced powers $\mathcal{P}^k g_n{}^\#$, and also the iterated Steenrod cyclic reduced powers of $g_n{}^\#$. Among other things he proves that $\mathcal{P}^k g_n{}^\# \neq 0$ for $2k \leqq n$.

Chapter III. Let K, a connected complex, be regularly imbedded in K^{Σ_q}. Let ι: $\pi_s(K) \to \pi_s(K^{\Sigma_q})$ be the injection homomorphism. $\alpha \in \pi_s(K)$ is killed by q-fold symmetrization of K if $\iota(\alpha)=0$. The author considers the following problem:

How many homotopy groups of S_n could be killed by symmetrization? He points out that symmetrization kills all the homotopy groups $\pi_i(S_2)$, since the q-fold symmetric product of S_2 is homeomorphic to the complex projective space of topological dimension $2q$. He proves that 2-fold symmetrization of S_n kills $\pi_{n+1}(S_n)$, $\pi_{n+2}(S_n)$, and the 2-primary subgroup of $\pi_{n+3}(S_n)$ for $n\geq 5$; 3-fold symmetrization kills the 3-primary subgroup of $\pi_{n+3}(S^n)$ for $n\geq 3$.

This chapter contains an explicit cellular decomposition of $S_n^{2_2}=S_n*S_n$ due to Steenrod. This decomposition is obtained inductively thanks to the following lemma: S_n*S_n is homeomorphic to the space obtained by first suspending $S_{n-1}*S_{n-1}$ and then adjoining a $2n$-cell to the suspension. The chapter ends with an application of symmetric products to the problem of secondary obstructions of maps of a complex into an $(n-1)$-connected space. As the author states, his main application in this regard is to the problem of secondary obstructions on fibre bundles [Liao, ibid. **60**, 146–191 (1954); MR **15**, 979].

The paper ends with an appendix where it is shown that every connected, simply-connected finite cellular complex is of the same homotopy type of some "reduced complex".

J. Adem (Mexico, D. F.).

de Carvalho, Carlos A. A. Sur les obstacles réduits de H. Hopf. II. C. R. Acad. Sci. Paris. **239**, 1574–1576 (1954).

The author shows that under certain conditions the results obtained in a previous paper [same C. R. **237**, 867–869 (1953); MR **15**, 458] can be made more precise.

J. Dugundji (Los Angeles, Calif.).

Nakaoka, Minoru, and Toda, Hirosi. On Jacobi identity for Whitehead products. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 1–13 (1954).

An important binary homotopy operation is the so-called Whitehead product which associates with elements $\alpha\in\pi_p(X)$ and $\beta\in\pi_q(X)$ the element $[\alpha,\beta]\in\pi_{p+q-1}(X)$. This operation introduced by J. H. C. Whitehead [Ann. of Math. (2) **42**, 409–428 (1941); MR **2**, 323] is known to be bilinear if $p>1$ and $q>1$, and to satisfy the usual commutativity rule of a graded algebra. The Whitehead product in general is not associative and for $\alpha\in\pi_p(X)$, $\beta\in\pi_q(X)$, $\gamma\in\pi_r(X)$, where p, q, and r are all >1, it has been conjectured that the following modified form of the Jacobi identity should hold:

$$(-1)^{pr}[[\alpha,\beta],\gamma]+(-1)^{qp}[[\beta,\gamma],\alpha]+(-1)^{rq}[[\gamma,\alpha],\beta]=0.$$

The authors give two proofs of this identity. One is by proving it in an universal space for Whitehead products, consisting of the union of three spheres with a single point in common. This proof uses the "generalized Whitehead products" of Blakers and Massey [ibid. **58**, 295–324 (1953); MR **15**, 731]. As the authors remark this proof can be modified so that the Jacobi identity also holds for some of the generalized Whitehead products of Blakers and Massey. The second proof is based on the theory of torus homotopy groups of R. H. Fox [ibid. **49**, 471–510 (1948); MR **10**, 260].

Using the Jacobi identity the authors show for $\alpha\in\pi_n(X)$, that $[\alpha,[\alpha,\alpha]]=0$ if n is odd, and $3[\alpha,[\alpha,\alpha]]=0$ if n is even. Also, if $T_i(\alpha)$ is any i-ple product (e.g., $T_4(\alpha)=[[\alpha,\alpha],[\alpha,\alpha]]$) then $T_i(\alpha)=0$ for $i\geq 4$. Now let $i_{2n}\in\pi_{2n}(S^{2n})$ be a generator. The authors show, with the use of Steenrod cubic operations, that if $[i_{2n},[i_{2n},i_{2n}]]=0$ then the 3-primary component of $\pi_{r+4n-1}(S^r)$ is non-trivial for $r\geq 2n+1$, and n is a power of 3.

[Reviewer's note: Independently several topologists have obtained proofs of the Jacobi identity. We give the following explicit references: H. Suzuki, Tôhoku Math. J. (2) **6**, 78–88 (1954); MR **16**, 276; W. S. Massey and H. Uehara, Volume commemorating the 70th birthday of Professor Lefschetz (to appear soon).] *J. Adem.*

Whitehead, George W. On mappings into group-like spaces. Comment. Math. Helv. **28**, 320–328 (1954).

Let G be a topological space in which there is defined a continuous multiplication $(x,y)\to x\cdot y$ and a continuous inversion $x\to x^{-1}$ such that the group axioms are satisfied "up to a homotopy." An important example of such a space is the space $\Omega(Y,y_0)$ of closed loops in a space Y based at the point $y_0\in Y$. The homotopy classes of continuous maps of any space X into G form a group, denoted by $\pi(X,G)$. Examples show that this group need not be abelian. However, under reasonable hypotheses on X and G, $\pi(X,G)$ is nilpotent [see p. 105 of H. Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937, for the definition]. If the Lusternik-Schnirelman category of X is c, then $c-1$ is an upper bound for the class of nilpotency of $\pi(X,G)$.

In the particular case where X is a product of spheres, more precise results are obtained. The class of nilpotency is less than or equal to the number of factors of X, and an explicit central chain for the group $\pi(X,G)$ is constructed. The successive factor groups are direct products of homotopy groups of G. In case $X=S^p\times S^q$ is the product of a p-sphere and a q-sphere, $\pi(X,G)$ is a central extension of $\pi_{p+q}(G)$ by $\pi_p(G)\times\pi_q(G)$, which is semi-split in the sense that each of the subgroups $\pi_p(G)$, $\pi_q(G)$ can be "lifted" to a subgroup of $\pi(X,G)$. If we take an element $\alpha\in\pi_p(G)$ and an element $\beta\in\pi_q(G)$, "lift" them into $\pi(X,G)$, and form their commutator, we obtain an element of the subgroup $\pi_{p+q}(G)$. This defines a bilinear map $(\alpha,\beta)\to\langle\alpha,\beta\rangle$ of $\pi_p(G)\times\pi_q(G)$ into $\pi_{p+q}(G)$. In case $G=\Omega(Y,y_0)$ is the space of loops in Y, then $\pi_n(G)$ is naturally isomorphic to $\pi_{n+1}(Y)$. This isomorphism transforms the operation $\langle\alpha,\beta\rangle$ into the Whitehead product in Y (except possibly for sign). By making a more detailed analysis of the case where $X=S^p\times S^q\times S^r$ is the product of three spheres, the author shows that the iterated commutators $\langle\alpha,\langle\beta,\gamma\rangle\rangle$ satisfy a Jacobi identity. It follows that the Whitehead products also satisfy a Jacobi identity [for another proof, see the paper reviewed above].

W. S. Massey (Princeton, N. J.).

Nakaoka, Minoru. On a theorem of Eilenberg-MacLane. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 31–39 (1954).

Let $\mathfrak{C}$ be a class of abelian groups in the sense of Serre [Ann. of Math. (2) **58**, 258–294 (1953); MR **15**, 548], having the properties:

(II$_B$) if $M\in\mathfrak{C}$, then $M\otimes N\in\mathfrak{C}$ for any abelian group N,

(III) if $M\in\mathfrak{C}$, then $H_i(M)\in\mathfrak{C}$, $i>0$.

The author proves that if Y, Y' are simply-connected spaces, such that

$$\pi_i(Y),\ \pi_i(Y')\in\mathfrak{C},\quad i<n,\quad n<i<q,\quad \pi_n(Y)\underset{\mathfrak{C}}{\cong}\pi_n(Y')$$

and if G is any coefficient group, then

$$H_i(Y;G)\underset{\mathfrak{C}}{\cong}H_i(Y';G)\underset{\mathfrak{C}}{\cong}H_i(\pi_n(Y),n;G),\quad i<q,$$

and

$$H_q(Y;G)/\Sigma_q(Y;G)\underset{\mathfrak{C}}{\cong}H_q(Y';G)/\Sigma_q(Y';G)\underset{\mathfrak{C}}{\cong}H_q(\pi_n(Y),n;G),$$

where Σ_q is the subgroup of spherical homology classes. This generalizes the well-known result of Eilenberg-MacLane (case $\mathfrak{C}=(0)$).

Two applications are made. If Y simply-connected, $\pi_i(Y)$ finite for all i, $\pi_{l_i}(Y)\otimes\pi_{l_j}(Y)=0$, $i, j=1, \cdots, m$, $i\neq j$, $\pi_i(Y)=0$, $i\neq l_1, \cdots, l_m$, then Y has the singular homotopy type of $K(\pi_{l_1}(Y), l_1)\times\cdots\times K(\pi_{l_m}(Y), l_m)$; but the author acknowledges that this result follows readily from the Postnikov theorems (see next review). Second, the author determines sufficient conditions on the Betti numbers of a simply-connected space Y to ensure that $\pi_i(Y)$ is infinite for at least two values of i. [Reviewer's note: with reference to footnote 3 on p. 35, it is known that the product of CW-complexes need not be a CW-complex].

P. J. Hilton (Cambridge, England).

Mizuno, Katuhiko. On the minimal complexes. J. Inst. Polytech. Osaka City Univ. Ser. A. **5**, 41–51 (1954).

The author describes a construction for the minimal complex of the singular complex of a simply-connected space Y, using the homotopy groups of Y and the Postnikov invariants [Dokl. Akad. Nauk SSSR (N.S.) **76**, 359–362, 789–791 (1951); MR **13**, 374, 375]. The construction is carried out explicitly if Y has vanishing homotopy groups in dimensions differing from n, q, $n<q$; the resulting minimal complex is called $K=K(\pi_n, \pi_q, k)$, where $\pi_n=\pi_n(Y)$, $\pi_q=\pi_q(Y)$ and k is the Eilenberg-MacLane-Postnikov invariant in $H^{q+1}(\pi_n, n; \pi_q)$. If $\pi_i(Y)=0$, $1\leqq i<n$, $n<i<q$, $q<i<r$, then the obstruction to extending the singular r-equivalence $K\rightarrow Y$ is an element of $H^{r+1}(K; \pi_r(Y))$ which is the next (non-vanishing) Postnikov invariant. The process of constructing K may be generalized, and the author thereby recovers the Postnikov theorem on the determination of the singular homotopy type of Y by its homotopy groups and Postnikov invariants. [Reviewer's note: the definition and role of the Postnikov invariants is further elucidated in a paper by J. F. Adams to appear in J. London Math. Soc.].

P. J. Hilton (Cambridge, England).

Nakaoka, Minoru. Transgression and the invariant k_n^{q+1}. Proc. Japan Acad. **30**, 363–368 (1954).

M. M. Postnikov [Doklady Akad. Nauk SSSR (N.S.) **76**, 359–362, 789–791 (1951); MR **13**, 374, 375] has defined certain cohomology invariants which, together with the homotopy groups, provide a complete characterization of homotopy type. In the formulation of J. H. C. Whitehead [Proc. London Math. Soc. (3) **3**, 385–416 (1953); MR **15**, 734] the Postnikov invariant $k^{q+1}(K)$, $q>1$, is defined as follows. Embed K in $\hat{K}$ such that $\pi_r(K)=\pi_r(K)$, $r<q$, $\pi_r(\hat{K})=0$, $r\geqq q$, be successively attaching cells of dimension $q+1, q+2, \cdots$. Then $k^{q+1}(K)$ is the class, in $H^{q+1}(\hat{K}; \pi_q(K))$, of the obstruction to extending the identity map $\hat{K}^q\rightarrow K$ over $\hat{K}^{q+1}$. Now the Cartan-Serre fibration gives a fibre-space E, of the homotopy type of K, over $\hat{K}$ such that the projection $E\rightarrow\hat{K}$ is equivalent to the injection $K\subseteq\hat{K}$. The injection of the fibre F into E induces isomorphisms $\pi_r(F)\cong\pi_r(E)$, $r\geqq q$, and $\pi_r(F)=0$, $r<q$. Thus there is a canonical isomorphism $H_q(F)\cong\pi_q(K)$, whence a distinguished element $b\in H^q(F; \pi_q(K))$. The author uses the techniques of G-duals [J. H. C. Whitehead, loc. cit.] to prove that $k^{q+1}(K)$ is precisely $-\tau(b)$, where τ: $H^q(F)\rightarrow H^{q+1}(\hat{K})$ is the transgression, valid if K is simply-connected.

In particular, if $\pi_r=\pi_r(K)=0$, $r\neq n$, q, $1<n<q$, then $\hat{K}=K(\pi_r, n)$, $F+K(\pi_q, q)$ and k^{q+1} is the Eilenberg-MacLane invariant $k_n^{q+1}\in H^{q+1}(\pi_n, n; \pi_q)$ determining the $(q+1)$-type of K.

P. J. Hilton (Cambridge, England).

Mizuno, Katuhiko. A proof for a theorem of M. Nakaoka. Proc. Japan Acad. **30**, 431–434 (1954).

Let (E, p, B) be a fibre-space (in the sense of Serre) such that E, B are simply-connected, $\pi_r(E)=0$, $r>q$, $\pi_r(B)=0$, $r\geqq q$, p_*: $\pi_r(E)\cong\pi_r(B)$, $r<q$. The author proves the result quoted in the preceding review, relating the Eilenberg-MacLane invariant k_n^{q+1} to transgression in the appropriate Cartan-Serre fibration, by taking a singular model, p': $S(E)\rightarrow S(B)$, of (E, p, B). He constructs a cross-section over the q-section of a minimal subcomplex, $M(B)$ of $S(B)$ and measures the obstruction to extending the cross-section over $M^{q+1}(B)$. The Postnikov invariants may be treated similarly. The notation of this paper is clarified in the paper reviewed second above.

P. J. Hilton. (Cambridge, England).

GEOMETRY

Sichardt, W. Ein Satz vom Kreis. Z. Angew. Math. Mech. **34**, 429–431 (1954).

The author proves the following theorem and applies it to a problem from hydraulics: If the circumference of a circle of radius A is divided into n equal parts and one of the dividing points joined with straight lines to each of the other $n-1$ points, then the product of the quotients of the secants by the radius is n.

Garrett, Milan Wayne. Solid angle subtended by a circular aperture. Rev. Sci. Instrum. **25**, 1208–1211 (1954).

Dias Agudo, Fernando Roldão. On some theorems from the geometry of quadrics. Ciência **4**, nos. 9–10, 59–67 (1954). (Portuguese. English summary)

Lomazzi, Luigi. Sulla generazione di alcune curve notevoli. Period. Mat. (4) **32**, 212–222 (1954).

Zacharias, Max. Bemerkung zu meiner Arbeit: "Die ebenen Konfigurationen (10_3)". Math. Nachr. **12**, 256 (1954).

See same Nachr. **6**, 129–144 (1951); MR **13**, 767.

Medek, Václav. Transformation of some nonlinear systems of conic systems. Mat.-Fyz. Sb. Slovensk. Akad. Vied Umenf **1**, 59–67 (1951). (Slovak. Russian summary)

L'auteur représente la conique d'un plan par le point de l'espace S_5 projectif à cinq dimensions (d'après le livre connu de Bertini, Einführung in die projektive Geometrie mehrdimensionaler Räume, Seidel, Wien, 1924) et use cette représentation pour obtenir les figures simples des systèmes non linéaires des coniques. P.e., on étudie la figure du système des coniques inscrites harmoniquement dans la conique donnée, ou la figure du système des coniques, qui ont en commun deux couples des points conjugués et deux couples des droites polaires conjuguées; il décrit aussi la figure du

système des coniques, qui passent par deux points et touchent deux droites données etc.

Le système des coniques est du degré n, si sa représentation dans S_5 est la courbe du degré n. On considère aussi la courbe polaire d'un point par rapport au système des coniques, c'est à dire, l'enveloppe des droites polaires du point par rapport aux coniques du système étudié. On a ensuite p.e. le théorème élémentaire: Si le système des coniques est du degré n, la courbe polaire d'un point général arbitraire par rapport au système a la classe n. Outre cela les propriétés des systèmes des coniques ne sont pas étudiées.

F. Vyčichlo (Prague).

Klingenberg, Wilhelm. Projektive und affine Ebenen mit Nachbarelementen. Math. Z. 60, 384–406 (1954).

If the axioms of a projective plane are altered, following Hjelmslev [Danske Vid. Selsk. Math.-Fys. Medd. 19, no. 12 (1942); MR 7, 472], to permit two distinct points to lie on two distinct lines, when this happens we say the points are neighboring points and the lines are neighboring lines. Remote points are joined by a unique line and remote lines intersect in a unique point. The relation of being neighboring is further assumed to be transitive and thus is easily seen to be an equivalence. Classes of neighboring points and lines then form the lines and points of an ordinary projective plane.

In this paper the main result is that, if in an affine plane with neighboring elements the theorem of Pappus is valid for remote points and lines, then the plane can be coordinatized by a Hjelmslev ring. This is an associative and commutative ring in which the elements without inverses form a radical, and modulo the radical, the ring is a field. Further, of two elements in the radical, one must be a multiple of the other. The author remarks that the known rings of this type are residue-class rings of rings with valuations.

Marshall Hall, Jr. (Columbus, Ohio).

Zappa, Guido. Sulle omologie dei piani $h-l$-transitivi e dei piani su quasicorpi. Ricerche Mat. 3, 35–39 (1954).

It has previously been shown by the author [Boll. Un. Mat. Ital. (3) 9, 16–24 (1954); MR 15, 818] that in a finite projective plane which admits every homology with center on a line h and axis a line l, the points not on either of these lines may be regarded as the elements of a doubly transitive group in which only the identity fixes two letters. Using this representation he shows that the roles of h and l may be reversed. He shows also that if the finite lines of a plane are given by $x=c$ and $y=xm+b$ with elements from a near field, then there exist all homologies with center X, the infinite point on the x-axis and axis of the homology any line $x=c$ through the infinite point Y on the y-axis. Here again the roles of X and Y may be interchanged.

Marshall Hall, Jr. (Columbus, Ohio).

Džavadov, M. A. Projective and non-Euclidian geometries over matrices. Dokl. Akad. Nauk SSSR (N.S.) 97, 769–772 (1954). (Russian)

This brief expository note is concerned with projective and noneuclidean geometry, the coordinates of points being m-rowed square matrices. One may consider three types according as the elements are real, complex or Clifford numbers $a+be$, where $e^2=1$. A collineation in a space over real matrices is defined by $y^i l=a_j{}^i x^j$, where l is a real nonsingular matrix as is also $\|a_j{}^i\|$. If the x^i are either complex or double, as the author calls them, and if $x=a+bi$ or $a+be$ and $\bar{x}=a-bi$ or $a-be$, then a collineation is defined by $y^i l=a_j{}^i\bar{x}^j$. Similarly, a correlation is defined by $lu^i=\tilde{a}^j a_{ij}$ in the real case and by $lu^i=\tilde{\bar{x}}^j a_j{}^i$ in the other two cases, where $\tilde{x}$ is the transpose of x. The cross ratio of four points on a line, x^i, y^i, $x^i+y^i a$, $x^i+y^i b$ is the matrix $W=ba^{-1}$. This obviously does not apply if a is singular but the author does not consider this possibility. Under collineations W undergoes a similarity transformation in case of real matrices and it becomes $k\bar{W}k^{-1}$ in the other two cases. One may define a linear congruence in a projective space (over real matrices) of $2n+1$ dimensions as the invariant lines of the collineation $'X^{2i}=X^{2i+1}$, $'X^{2i+1}=X^{2i}$ and in the other two cases using conjugates. From this one obtains the relations between the collineation groups of the space and of the congruence. These various spaces may be considered as subspaces over matrices whose elements are pseudoquaternions, $\alpha=a+bi+ce+df$, where $f=ie=-ei$. By introducing an absolute polarity and the pole and polar with respect to this absolute, one is able to study the metric properties of such spaces.

M. S. Knebelman (Pullman, Wash.).

Ganea, Tudor. The distance between closed subsets defined by D. Pompeiu. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 5, 25–28 (1954). (Romanian. Russian and English summaries)

Expository paper concerning the hyperspace of a space metrized by D. Pompeiu's formula.

Author's summary.

Convex Domains, Extremal Problems, Integral Geometry

Sz.-Nagy, Béla. Ein Satz über Parallelverschiebungen konvexer Körper. Acta Sci. Math. Szeged 15, 169–177 (1954).

A collection C of sets is said to have the binary intersection property provided every subcollection of C, each two members of which intersect, has non-empty intersection. In a paper on extension of linear transformations [Trans. Amer. Math. Soc. 68, 28–46 (1950); MR 11, 369], Nachbin proved: (1) A symmetric convex body in E^n is a parallelotope if and only if the collection of all its homothetic images has the binary intersection property. The present paper establishes: (2) A convex body in E^n is a parallelotope if the collection of all its translates has the binary intersection property. An equivalent formulation, noted by G. Hajós, is: (2') A convex body K in E^n is a parallelotope if every n-simplex $[Q_0\cdots Q_n]$, each of whose sides $[Q_iQ_j]$ is contained in some translate of K, is itself contained in some translate of K.

The principal lemma is: (L) If K is a convex body in E^n ($n\geqq 2$) and is not a parallelotope, there are an integer m with $3\leqq m\leqq n+1$, regular boundary points P_i of K, and positive numbers a_i such that with n_i denoting the unit outward normal to K at P_i, then $\sum_1^m a_i n_i=0$ and always $n_j\neq -n_k$. (2) is proved by showing that for a sufficiently small positive t the collection $\{K-P_i-tn_i\}_{i=1}^m$ of translates of K has the binary intersection property but empty intersection.

Reviewer's comments: (i) (2) is intermediate between (1) and a proposition stated without proof by Nachbin [loc. cit., p. 43]: (3) A compact subset of E^n with nonempty interior is a parallelotope if the collection of all its translates has the binary intersection property. The author's statement of (3) [footnote, p. 172] is not quite accurate. (ii) In order to

conclude that the set R of regular points in the boundary B of a convex body K is dense in B, the author uses a theorem on the measure of $B \setminus R$. However, it is clear that if $p \in K \setminus B$ and q is a point of B nearest to p, then $q \in R$, and that the set of points q so obtained, for p ranging over $K \setminus B$, is dense in B. *V. L. Klee* (Seattle, Wash.).

Hlawka, Edmund. Über eine Klasse von mehrfachen Integralen. Abh. Math. Sem. Univ. Hamburg **18**, 53–69 (1952).

A discussion, largely without proofs, of the author's work about integrals on convex bodies [Monatsh. Math. **54**, 1–36, 81–99 (1950); **55**, 105–137 (1951); MR **12**, 197, 198; **13**, 154]. *J. W. S. Cassels* (Cambridge, England).

Schoenberg, I. J. An isoperimetric inequality for closed curves convex in even-dimensional Euclidean spaces. Acta Math. **91**, 143–164 (1954).

A closed curve in E_{2n} is called convex provided that it never crosses a hyperplane more than $2n$ times. The main result of the paper is the following noteworthy theorem: Let C be a closed curve convex in E_{2n} and let L denote its length. Let $K=K(C)$ be the convex hull of C and let $V=V(K)$ denote the $2n$-dimensional volume of K. Then the following inequality holds:

$$L^{2n} \geq (2\pi n)^n n!(2n)! V(K)$$

with the equality sign if and only if the curve C agrees up to a rigid motion and a similitude, followed perhaps by a reflexion, with the curve defined by

$$x_1=\cos t,\quad x_3=\tfrac{1}{2}\cos 2t,\ \cdots,\ x_{2n-1}=\frac{1}{n}\cos nt,$$

$$x_2=\sin t,\quad x_4=\tfrac{1}{2}\sin 2t,\ \cdots,\ x_{2n}=\frac{1}{n}\sin nt \quad (0\leq t\leq 2\pi).$$

The proof uses some inequalities on determinants and matrices which are interesting by themselves but which are too long to be summarized here. *L. A. Santaló.*

Santaló, L. A. Generalization of a geometric inequality of Feller. Rev. Un. Mat. Argentina **16**, 78–81 (1954). (Spanish. English summary)

A very short proof is given for the following result of Ueno, Hombu and Naito [Mem. Fac. Sci. Kyûsyû Univ. A. **6**, 97–106 (1951); MR **13**, 864]. Let D be a domain in a sphere of radius α in an n-dimensional space of constant curvature $K=k^2$. If any r-dimensional linear subspace $(1\leq r\leq n-1)$ intersects D in a set of measure at most δ, then the measure of D is at most

$$\delta\omega_{n-r-1}k^{r+1-n}\int_0^\alpha \cos^r ku \sin^{n-r-1} ku\, du,$$

where $\omega_i=\Gamma[\frac{1}{2}(i+1)]\frac{1}{2}\pi^{-(i+1)/2}$. [For $r=1$ and $K=0$ this result was previously obtained by Feller, Duke Math. J. **9**, 885–892 (1942); MR **4**, 168.] The case $r=1$ (with 1-dimensional subspace = geodesic) is extended to arbitrary Riemann spaces: the measure of D is at most $\delta(n-1)^{-1}\omega_{n-2}\omega_{n-1}^{-1}F$, where F is the area of the hypersurface bounding D.

H. Busemann (Göttingen).

Huber, Alfred. On the isoperimetric inequality on surfaces of variable Gaussian curvature. Ann. of Math. (2) **60**, 237–247 (1954).

Generalizing the isoperimetric inequality $L^2\geq 4\pi A$, which characterizes surfaces of non-positive Gaussian curvature K, and the inequality $L^2\geq 2A(2\pi-\iint_D K df)$, where df is the surface element of the domain D in question, which Fiala [Comment. Math. Helv. **13**, 293–346 (1941); MR **3**, 301] has shown to hold on regular surfaces with $K\geq 0$, the author shows that the inequality $L^2\geq 2A(2\pi-\iint_D K^+ df)$, where $K^+=\max[K, 0]$, holds on sufficiently smooth abstract surfaces. The proof is based on inequalities for differences of subharmonic functions which the author establishes, and which are of interest in themselves. It is shown in conclusion that the foregoing inequality of Fiala is characteristic of surfaces of nonnegative Gaussian curvature.

E. F. Beckenbach (Los Angeles, Calif.).

Corrsin, Stanley. A measure of the area of a homogeneous random surface in space. Quart. Appl. Math. **12**, 404–408 (1955).

Let us consider the n-dimensional space homogeneously inscribed with p-dimensional varieties of arbitrary shape and let V be the average p-dimensional area per unit volume. If n is the average number of cuts per unit q-dimensional area made by a random linear q-dimensional variety $(p+q=n)$ in the space, then standard methods of integral geometry lead to the formula $n=(2O_n/O_pO_q)V$, where $O_i=2\pi^{(i+1)/2}/\Gamma((i+1)/2)$. In the present paper the author gives a simple and direct proof of that formula for the cases $n=2$, $p=1$, $q=1$ and $n=3$, $p=2$, $q=1$. *L. A. Santaló.*

Algebraic Geometry

Bonera, Piero. Sui gruppi di livello del cubo della curvatura proiettiva per la cubica ellittica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 333–345 (1953).

Bonera, Piero. Sui gruppi di livello del cubo della curvatura proiettiva per la cubica nodata. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 346–350 (1953).

In these papers, the author studies the linear pencil defined on a cubic curve by the sets of constant level for the cube of the projective curvature. Particular sets are examined and it is shown that the linear pencil is of order 216 for elliptic cubics and of order 24 for nodal cubics.

G. B. Huff (Athens, Ga.).

Marchionna, Ermanno. Serie lineari complete su una curva gobba dotata di punti multipli ed intersezione totale di due superficie. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 184–222 (1953).

The author considers an irreducible curve Γ^* which is the complete intersection of $d-1$ primals in S_d. The orders of these are $\mu_1, \cdots, \mu_{d-1}$, and they are supposed to have multiplicity $h_1, \cdots, h_{d-1}$ in a point O^*, such that Γ^* has only simple branches with distinct tangents in O^*. The bulk of the paper is devoted to the case $d=3$, but almost all the results obtained are ultimately established also in the general case. The principal of these are:

The complete canonical series is traced on Γ^* by primals of order

$$m=\mu_1+\cdots+\mu_{d-1}-d-1$$

with an $(s-1)$-ple point at O^*, where

$$s=h_1+\cdots+h_{d-1}-d+1$$

and touching every branch of Γ^* in O^*. It is also traced by primals of order m with an s-ple base point in O^*, provided

none of the primals whose complete intersection is Γ^* is a cone.

A sufficient condition for primals of order l, having with each branch of Γ^* at O^* i intersections coinciding in O^*, to trace residually a complete series on Γ^* is (a) $i\geqq s$ if $l>m$; (b) $i\geqq s-j$ if $l=m+1-j$ $(j\leqq s)$; (c) $i\geqq 0$ if $l\leqq m+1-s$. In cases (a), (c) the condition is also necessary.

Γ^* is normal except when one of the intersecant primals is a monoid and the rest are all cones of vertex O^*.

Some attention is also given to the case in which Γ^*, instead of the singularity O^*, has k double points. Here it is shown that l-ic primals through all the double points if $l>m$, or through at least $k-j$ of them if $l=m-1-j$, trace residually a complete series. The curve is normal if $k<m$.
P. Du Val (London).

Marchionna, Ermanno. Costruzione di una funzione algebrica di due o più variabili avente un'assegnata varietà di diramazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 127–147 (1953).

In a former paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 170–177 (1951); MR **13**, 974] the author gave necessary and sufficient conditions for an irreducible plane curve Γ of order n with δ nodes and k cusps to be the branch curve of a $(\mu-h)$-ple plane projection of a μ-ic surface from an h-ple point, which are here restated as follows:

$$n=\mu(\mu-1)-h(h+1),$$
$$\delta=\tfrac{1}{2}n[(\mu-2)(\mu-3)-h(h+1)]-2h(h+1)(\mu-h-3),$$
$$k=\mu(\mu-1)(\mu-2)-h(h+1)(h+2),$$

and there exist (a) a curve e_l of order
$$l=(\mu-1)(\mu-2)-h(h+1)$$
adjoint to Γ and meeting it residually in a set of $h(h+1)$ points T each counted $2(\mu-h-2)$ times; (b) a curve e_{l+1} of order $l+1$, not consisting of e_l and a line adjoint to Γ and meeting it residually in the set T counted $2(\mu-h)-5$ times and in further points none of which is included in T.

The present paper gives a method of actually constructing the multiple plane as an algebraic irrationality over the coordinates in the plane, i.e. of constructing the surface of which it is the projection, given the equations of Γ, e_l, e_{l+1}, and a curve b_h of order h passing through the set T. The method is something like an Euclidean algorithm applied to the functions concerned as functions of z, and beginning at the end. It is carried out in detail as illustration in the case $\mu=9$, $h=4$.
P. Du Val (London).

Marchionna, Ermanno. Precisazioni su un'estensione di un teorema di Halphen. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 101–110 (1953).

A counter-example to Theorem II of an earlier paper of the author [same Rend. (3) **14**(83), 137–158 (1950); MR **13**, 974] having been shewn him by Andreotti, he amends the theorem as follows: If a space-curve Γ^* of order $\mu\nu$ $(\mu\geqq\nu)$ with an hk-ple point O^* projects from a general point into a plane curve Γ with hk-ple point O, and with $\tfrac{1}{2}[\mu\nu(\mu-1)(\nu-1)-hk(h-1)(k-1)]$ double points which are its complete residual intersection with a curve of order $(\mu-1)(\nu-1)$ which is $(h-1)(k-1)$-ple at O, then Γ^* lies on a surface B of order ν, with multiplicity in O^* between h and k inclusive, and on a linear system of freedom $\binom{\mu-\nu+3}{3}$ of surfaces A of order μ, with multiplicity in O^* which is the lesser of h, k, and if $h\neq k$ all having in O^* the same tangent cone. Γ^* is the complete intersection of B with a general A, unless it lies on some surface of order $<\nu$, in which case B is reducible and Γ^* is not in general a complete intersection.

He analyses the proof of the original theorem, which consisted in an algebraic method of constructing the surfaces A, B, and shows how if certain polynomials have a common factor, the method gives a surface of lower order than ν containing the curve.
P. Du Val (London).

Tibiletti, Cesarina. Precisazioni sulla dimostrazione di un teorema di Halphen. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 80–85 (1953).

This consists of exactly the same work as the paper reviewed above, provoked by the same counter-example of Andreotti, except that neither the author's original statement [Ann. Mat. Pura Appl. (4) **31**, 69–81 (1950); MR **13**, 62] nor the present amendment take account of a possible singularity of the space-curve.
P. Du Val.

Tibiletti, Cesarina. Determinazione algebrico geometrica di piani tripli e piani quadrupli con la stessa curva di diramazione. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 86–100 (1953).

It is familiar that every fourfold plane has the same branch curve as some triple plane, since every quartic in one variable has a cubic resolvant whose discriminant is the same. This paper considers the converse questions, given a triple plane, is there a fourfold plane with the same branch curve? It is shown that if the triple plane is represented by $f=0$, where f is a cubic in z whose coefficients are of any order in the non-homogeneous coordinates in the plane, i.e. $f=0$ is a surface whose projection from the point at infinity on the z axis is the triple plane in question, the problem of constructing a fourfold plane with the same branch curve reduces to the construction of an irreducible two-sheeted surface covering $f=0$ without branch curve. Two such double surfaces which are birationally distinct give birationally distinct fourfold planes. The number of such two sheeted surfaces, birationally distinct, is known [A. Comessatti, Rend. Sem. Mat. Univ. Padova **1**, 1–45 (1930)] to be equal to the number of virtual half-zero linear systems on $f=0$. (The point is not stressed, but it appears that the two sheeted surface without branch curves may have isolated branch points in singularities of $f=0$, and similarly the half-zero systems need not remain so when neighbourhoods of singularities are taken into account.)
P. Du Val

Marchionna Tibiletti, Cesarina. Sostituzioni legate ad una curva di diramazione che possa degenerare in parti doppie. Ann. Mat. Pura Appl. (4) **37**, 333–346 (1954).

Chisini a ramené l'étude des courbes de diramation f des plans multiples représentatifs des surfaces algébriques à un type décomposé, formé d'une courbe double $\bar{f}$, dont les composantes irréductibles f_r n'ayant pour singularités que des points doubles ordinaires, sont telles que trois ne passent pas par un point et que les intersections de deux d'entre elles sont simples. A la courbe f qui se réduit à $2\bar{f}$ on peut associer des lacets du plan $x=0$, entourant les points de diramation et dont le parcours entraîne l'échange (i, k) sur deux déterminations de la fonction $z(0, y)$. Dans le passage à la forme limite, les diramations se rangent en couples, et au point de diramation correspondant on peut associer une substitution $S=(i, k)(i', k')$. L'étude de l'A. a pour but d'établir par l'utilisation des tresses les relations qui inter-

viennent entre ces substitutions. L'A. établit ainsi: 1) Sur une même composante f_r les diramations ont la même substitution. 2) Un couple de composantes f_1 et f_2 se coupant en un point limite de trois cuspides admet deux types de substitutions

A) $S_1=(I,2)(1,2)$, $S_2=(I,3)(1,3)$

B) $S_1=(I,2)(3,4)$, $S_2=(I,3)(2,4)$.

3) Un couple de composantes se coupant en un point limite de quatre points doubles correspond à des substitutions appartenant à 9 types si les points doubles sont essentiels, à 11 types s'ils ne sont pas tous essentiels. De ces conditions on déduit que la courbe n'est de diramation pour un plan quadruple que si une intersection limite de trois cuspides admet une substitution du type B. Si le plan n'est pas quadruple ($n>4$) les échanges (i, k) et (i', k') liés a une composante sont égaux. *B. d'Orgeval* (Alger).

Fava, Franco. Le reti di coniche dotate di cayleyana riducibile. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 46–54 (1954).

Fava, Franco. Invariante di Mehmke-Segre e reti di coniche. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 161–169 (1954).

Vaona, Guido. Le trasformazioni fra piani che posseggono infinite coppie di curve omografiche od affini. Boll. Un. Mat. Ital. (3) 9, 250–261 (1954).

L'auteur étudie certaines questions se rattachant à la considération des transformations ponctuelles entre deux plans π et π', pour lesquelles il existe des couples de courbes, situées respectivement dans π et π', et entre lesquelles la transformation ponctuelle envisagée établit une homographie, ou plus particulièrement une affinité. Suivant une suggestion de E. Čech, il envisage plus spécialement les transformations pour lequelles il existe ∞^h couples de courbes répondant à la question, en portant spécialement l'attention sur le problème de la recherche de la valeur maximum (h_m) que peut prendre h sans que la transformation se réduise à une homographie (ou à une affinité), et sur celui de la détermination des transformations qui possèdent ∞^{h_m} couples de courbes homographiques ou affines correspondantes. En ce qui concerne le premier problème l'auteur montre que l'on ne peut avoir ∞^h couples de courbes correspondantes homographiques ou affines que si $h\leqq 3$ $(h_m=3)$. Au sujet de la détermination des transformations ellesmêmes, il établit le résultat suivant, se rapportant au seul cas affine: Si on suppose π et π' contenus dans un même espace S_3, et si l'on se donne une surface fixe F de S_3 et deux points S_∞ et S_∞' de S_3 n'appartenant respectivement pas à π et π', les points obtenus en projetant une même point de F sur π et π' à partir de S_∞ et S_∞' respectivement, établissent entre π et π' la transformation la plus générale du type envisageé. L'auteur termine son travail par l'examen des particularités que doit présenter le système $[F, S_\infty, S_\infty']$ pour que les transformations entre π et π' fournies par la construction précédente soient de 2ème ou de 3ème espèce. *P. Vincensini* (Marseille).

Goddard, L. S. Cremona transformations in the geometry of matrices. Rend. Circ. Mat. Palermo (2) 2 (1953), 393–413 (1954).

The first part of the paper deals with the Cremona transformations between two $[n^2-1]$'s: Σ_1, Σ_2, homogeneous coordinates in which are taken to be the elements of square matrices X, Y, defined by

$$Y=\operatorname{adj}(X), \quad \sigma X=\operatorname{adj}(Y),$$

where of course $\sigma=|X|^{n-2}$. In Σ_i $(i=1, 2)$ are loci $K_i^{(r)}$ $(r=1, \cdots, n-1)$ defined by the matrix being of rank $\leqq r$, and $K_i^{(r)}$ is generated by $[r^2-1]$'s $t_i^{(r)}$. The primes of Σ_2 correspond to $(n-1)$-ic primals through $K_1^{(n-2)}$, and there is a one-one correspondence between the spaces $t_1^{(r)}$, $t_2^{(n-r)}$ such that every point of $t_1^{(r)}$ corresponds to every point of the corresponding $t_2^{(n-r)}$. Cases $n=3, 4$ are briefly examined, and a method given for finding the order of the V_k in Σ_1 corresponding to a general $[k]$ of Σ_1. The transformations induced between the subspaces Σ_1', Σ_2' corresponding to symmetric matrices, and (for even values of n) Σ_1'', Σ_2'' corresponding to skew symmetric matrices are also studied—the latter being of order only $\frac{1}{2}(n-2)$ instead of $n-1$, owing to the common factor in the minors of a skew-symmetric matrix.

We then take two $[n]$'s: Σ_1^*, Σ_2^* in which the coordinates are λ and the components of X, μ and the components of Y; and consider the transformation

$$\mu=|\lambda E+X|, \quad Y=(\lambda E-X)\operatorname{adj}(\lambda E+X),$$

which is likewise symmetrical, as it involves

$$\mu E+Y=2\lambda \operatorname{adj}(\lambda E+X).$$

In the prime $\bar{\omega}$ in Σ_1, given by $\lambda=0$, are the loci $K_1^{(r)}$ as before; and the locus $D_1^{(r)}$ defined by $\lambda E+X$ being of rank $\leqq r$ is the cone projecting $K_1^{(r)}$ from the point $D_1^{(0)}$ or $(-1, E)$. Primes of Σ_2 correspond to n-ic monoids in Σ_1, with $(n-1)$-ple point $D_1^{(0)}$ and simple base loci $K_1^{(n-1)}$, $D_1^{(n-2)}$; and there is a similar correspondence to that in the former case between a space $t_1^{(r)}$ and a $[(n-r)^2]$ $t_2^{*(n-r)}$ projecting a $t_2^{(n-r)}$ from $D_2^{(0)}$. Cases $n=2$, 3 are studied briefly, as is the case of the symmetric matrices (but not the skew-symmetric, presumably because $\lambda E+X$ is never skew symmetric for $\lambda\neq 0$). *P. Du Val* (London).

Balsimelli, Pio. Studio di una trasformazione birazionale dell'S_3 complesso determinata da una trasformazione quadratica biduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) 20 (1953), 273–278 (1954).

Fadini, Angelo. Studio di una trasformazione cremoniana dell'S_3 dedotta da una trasformazione quadratica dell'S_1 triduale avente i tre punti eccezionali coincidenti. Rend. Accad. Sci. Fis. Mat. Napoli (4) 20 (1953), 324–334 (1954).

Spampinato, Nicolò. Le falde t-dimensionali analitiche, algebriche ed unirazionali di un S_r supercomplesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) 20 (1953), 147–152 (1954).

Spampinato, Nicolò. Le superficie iperellittiche dell'S_3 biduale e la loro rappresentazione con varietà V_4 dell'S_7 complesso. Rend. Mat. e Appl. (5) 14, 164–180 (1954).

Spampinato, Nicolò. Varietà determinata da una terna ordinata di ipersuperficie dell'S_r complesso nell'S_{3r+3}. Ricerche Mat. 3, 13–30 (1954).

A detailed study of certain varieties V_{3r-1} in complex projective S_{3r-2}, motivated by the author's study of hypersurfaces in tri-dual (or tri-potential) S_r. *G. B. Huff.*

Godeaux, Lucien. Sur les surfaces algébriques touchant un plan le long d'une droite. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 1194–1198 (1954).

Dans cette note, d'un caractère assez élémentaire, nous précisions les singularités d'une surface algébrique aux points d'une de ses droites le long de laquelle elle a un plan tangent fixe. *Résume de l'auteur.*

Roth, Leonard. Improperly Abelian varieties. Rend. Sem. Mat. Univ. Padova **23**, 277–289 (1954).

The author continues his study of pseudo-Abelian varieties initiated in a previous paper [Proc. Cambridge Philos. Soc. **50**, 360–371 (1954); MR **16**, 66]. He proves that any Abelian variety W_p having some plurigenus greater than zero, which maps an involution of superficial irregularity q $(0<q<p)$ on a Picard variety V_p, is pseudo-Abelian of type q; and deduces that W_p is improperly Abelian, being representable parametrically in terms of Abelian functions of genus q and Abelian functions of genus $p-q$. It is shown that the classification of improper Abelian varieties of dimension p depends on the determination of all species of improperly Abelian varieties of lower dimension which map involutions on Picard varieties generated by finite groups of automorphisms (including these species which have all their plurigenera equal to zero).

The author concludes with a brief account of para-Abelian varieties. Such a variety W_p is a para-Abelian variety of type q if the following conditions are satisfied: (i) W_p contains a congruence $\{V_q\}$ of Picard varieties V_q, the irreducible members of which are non-singular and birationally equivalent; (ii) W_p contains a second congruence $\{V_{p-q}\}$, which is of Abelian type, whose irreducible members are non-singular and birationally equivalent; (iii) if $\bar{V}_q^*$ and $\bar{V}^*_{p-q}$ are birationally equivalent to $\{V_{p-q}\}$ and $\{V_q\}$ respectively, and if $\alpha=(V_q\cdot V_{p-q})$, then W_p can be mapped on the α-ple variety $\bar{W}_p^*=\bar{V}_q^*\times\bar{V}^*_{p-q}$ in such a way that the branch locus is generated by varieties V_q^* and V^*_{p-q} corresponding respectively to members of $\{V_q\}$ and $\{V_{p-q}\}$, each counted a certain number of times, and this mapping has no exceptional features other than those which result from these assumptions. Such a W_p does not admit the group of automorphisms which characterizes a pseudo-Abelian variety, but possesses a number of similar properties: e.g. the superficial irregularity of W_p is at least equal to the sum of the irregularities of the congruences $\{V_q\}$, $\{V_{p-q}\}$; and the virtual canonical system belongs to $\{V_q\}$, and contains (to multiplicity $s-1$) every variety generated by the reducible varieties V_{q_s} of $\{V_q\}$ which are $(s-1)$-ple components of the coincidence locus of the involution I_α of sets $(V_q\cdot V_{p-q})$. *J. A. Todd* (Cambridge, England).

Yûjôbô, Zuiman. On the continuous systems of $n-1$-dimensional algebraic varieties in an n-dimensional algebraic variety. Comment. Math. Univ. St. Paul. **3**, 37–42; corrections, 95 (1954).

La démonstration de l'existence des systèmes continus dont il est question dans le titre est une extension aux variétés à $n>2$ dimensions de propriétés bien connues des surfaces algébriques. En indiquant par V^n une variété algébriques à n dimensions $(n>2)$, l'A. se sert des sommes abéliennes on "fonctions normales" de Poincaré relatives à certaines courbes sections planes de V^n pour obtenir un critère d'équivalence linéaire de deux variétés à $n-1$ dimensions appartenant à V_n; et de là, en appliquant le théorème d'existence de Lefschetz et Poincaré, il déduit l'existence sur V^n d'un système continu de variétés à $n-1$ dimensions composé de ∞^q systèmes linéaires, où q représente l'irrégularité des surfaces sections de V^n par des espaces à trois dimensions (on suppose que ces surfaces possèdent seulement des singularités ordinaires). *E. Togliatti* (Gênes).

Nollet, Louis. Système canonique, plurigenres, diviseurs du zéro et nombre-base des surfaces algébriques régulières ayant un faisceau de courbes elliptiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 914–937 (1954).

En s'appuyant sur un travail précédent [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8° **28**, no. 3 (1953); MR **15**, 342], l'A. s'occupe ici des surfaces algébriques régulières contenant un faisceau $\mathfrak{F}$ de courbes elliptiques généralement irréductibles. Le point de départ est la considération des courbes réductibles contenues dans le faisceau $\mathfrak{F}$. En général, si $C=h_1C_1+h_2C_2+\cdots+h_tC_t$ est une courbe réductible de composantes irréductibles $C_1, \cdots, C_t$, l'A. dit que C est de longueur t, de support $C_1+\cdots+C_t$; il appelle diviseur de C le plus grand commun diviseur des entiers h. Une courbe irréductible est de diviseur 1. Cela posé, soit $e_1E_1, \cdots, e_sE_s$, $E_{s+1}, \cdots, E_r$ les courbes réductibles du faisceau $\mathfrak{F}$, où les lettres E_i signifient des courbes de diviseur 1; les courbes fondamentales du faisceau $\mathfrak{F}$ ont un degré negatif ou nul; l'A. donne avant tout, par deux procédés différents, une condition nécessaire et suffisante pour qu'une telle courbe fondamentale soit de degré zéro. Il en déduit: l'expression générale des divisieurs du zéro sur la surface donnée; l'expression du système canonique K en fonction de la courbe générale E de $\mathfrak{F}$ et des courbes $E_1, \cdots, E_s$; la valeur des plurigenres de la surface en fonction de p_g, de s et des nombres e_i; la valeur de l'invariant de Zeuthen-Segre à l'aide des longueurs des éléments réductibles de $\mathfrak{F}$. Le dernier chapitre s'occupe de la base des courbes de la surface donnée: on peut toujours former une base avec des courbes fondamentales de $\mathfrak{F}$ et des m-sécantes de $\mathfrak{F}$, où m est le déterminant de la surface; on en déduit pour le nombre-base une borne inférieure en fonction de la longueur des éléments réductibles de $\mathfrak{F}$. *E. G. Togliatti* (Gênes).

Segre, Beniamino. Dilatazioni e varietà canoniche sulle varietà algebriche. Ann. Mat. Pura Appl. (4) **37**, 139–155 (1954).

Let P' be a non-singular sub-variety, of dimension p, of a non-singular variety V' of dimension v; and let V be the transform of V' by means of a dilation with base P', and P the variety on V, of dimension $v-1$, which represents the neighborhood of P'. The author establishes the formula

$$P'_{V',i}=(-1)^{v-p+i-1}(P^{(v-p+i)})'_V,$$

in the notation of a previous paper [same Ann. (4) **35**, 1–127 (1953); MR **15**, 822]. The formula was wrongly stated in the earlier paper. As an application, a solution is given, in a number of cases, of the following problem: given a non-singular variety V' and a sub-variety P' of it, to determine, from a knowledge of the canonical system of V' and P', the canonical systems on the variety V obtained from V' by a dilation of base P'. *W. V. D. Hodge.*

Differential Geometry

***Biernacki, Mieczysław. Geometria różniczkowa. Część pierwsza. [Differential geometry. Part one.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1954. 240 pp. zł. 21.35.

This is the first volume of a planned two-volume textbook on classical differential geometry. The first volume is de-

voted to curves. The first chapter deals with plane curves. The second chapter contains the theory of curves in three dimensions. The first chapter goes as far as theory of contact, envelopes, evolutes and evolvents. A special section is devoted to epi- and hypocycloids. In the second chapter the author often uses vector calculus. After discussing classical elements of the theory of curves, he applies them to the Bertrand curves. Although this volume is chiefly devoted to curves, the author considers also the relationship of surfaces and curves (envelopes, developable surfaces, contact of curves and surfaces and so on). The number of well chosen problems with solutions (279) makes the book very valuable to teachers. *V. Hlavatý* (Bloomington, Ind.).

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★Julia, Gaston. Cours de géométrie infinitésimale. Deuxième fascicule. Cinématique et géométrie cinématique. Première partie: Généralités. 2ème éd. Gauthier-Villars, Paris, 1955. 80 pp. 1500 francs.

The first "fascicule" of this second edition appeared in 1953 [MR 15, 352] and dealt with vectors and tensors. This second "fascicule" has a chapter on the cinematics of a point, another on the cinematics of a solid body, and a chapter on the composition of motions. There are some interesting applications, as the remark on p. 53 that if a fixed system S_2 moves with respect to another fixed system S_1 such that a point M of S_2 describes a plane trajectory in a plane π linked to S_1, then in the inverse movement of S_1 with respect to S_2 this plane π passes through a fixed point of S_2. On pp. 59–67 we find a discussion of surfaces in the form $f(r_1, r_2, r_3)=0$, where the r_i are the distances of a point on the surface to three given fixed surfaces; this approach is connected with the names of Poinsot and Roberval. The "fascicule" ends with a chapter on the moving trihedron. For exercises the author usually refers to his own "Exercices de géométrie infinitésimale" [2 fasc., Gauthier-Villars, Paris, 1944, 1952] and to the "Exercices de mécanique" [t. I, fasc. 2, 2ème éd., ibid., 1951; MR 12, 449] by H. Beghin and the author. *D. J. Struik* (Cambridge, Mass.).

Laugwitz, Detlef. Differentialgeometrie ohne Dimensionsaxiom. I. Tensoren auf lokal-linearen Räumen. Math. Z. 61, 100–118 (1954).

The author is concerned with the creation of a theory of differential geometry in a form not limited to finite-dimensional spaces, but consistent with the classical theory in case a finite-dimensionality condition is imposed. He attaches great importance to the development of an adequate and convenient notation. In his opinion the work of earlier writers on differential geometry in function-spaces and abstract infinite-dimensional spaces [G. Kowalewski, C. R. Acad. Sci. Paris **151**, 1338–1340 (1910); G. Vitali, Geometria nello spazio hilbertiano, Zanichelli, Bologna, 1929; A. D. Michal, Bull. Amer. Math. Soc. **45**, 529–563 (1939); MR 1, 29; M. Kerner, Compositio Math. **4**, 308–341 (1937)] is burdened by overly cumbersome notation which makes formulas too long, hinders the perception of geometrical content in the work, and impedes the discovery of new results. The author essays to improve matters by the introduction of various notational conventions.

Let B be a real Banach space. Any symbol for an element of B will be affected with a small Latin letter as superscript. The superscript identifies the element as belonging to B, but distinction between different elements of B is made by changing the basal letter, so that $x^a=x^b$. Continuous linear functionals on B are denoted by letters affected with small Latin letters as subscripts. The Banach space of such functionals is denoted by B_1. Likewise, elements of the space B_2 of continuous bilinear functionals on $B\times B$ are affected with two small Latin letters as subscripts. If $t_{ik}\in B_2$, its value at the pair $(x^i, y^k)\in B\times B$ is denoted by $t_{ik}x^iy^k$. The corresponding "quadratic form" is denoted by $t_{ik}x^ix^k$. There is a convention about free and bound indices, but no summation convention. In the case of the bilinear functional t_{ik}, $t_{ik}y^k$ is an element of B_1 whose value at x^i is $t_{ik}x^iy^k$; here y^k is regarded as fixed. Different sets of characteristic indices are used for different Banach spaces. If $a, b, \cdots$ are characteristic of B and $\alpha, \beta, \cdots$ are characteristic of C, a bounded linear operator mapping B into C is affected with two indices, one upper and one lower, thus: A_a^α.

A topological space M is called B-locally linear if there exists an open covering of M composed of subsets of M each of which is homeomorphic to an open set in B. Points of B are used as coordinates of points in M, and the local mappings of parts of B into B induced by the overlapping of the covering sets in M are required to have continuous Fréchet differentials up to some specified order. Contravariant vectors at a point are represented by elements of B; covariant vectors are represented by elements of B_1. The laws of transformation are expressed by means of Fréchet differentials. An m-fold contravariant, n-fold covariant tensor is represented, in a given coordinate system, by a multilinear functional $T^{i_1\cdots i_m}_{k_1\cdots k_n}$ on $B^m\times B_1{}^n$, where B^m is the m-fold product $B\times\cdots\times B$, and $B_1{}^n$ has a similar meaning. The linear functionals on B_1 which occur are limited to those represented by elements of B. Laugwitz's definition of covariant vectors is (in general) different from that of Michal. Laugwitz's work also differs from the work of Michal and Kerner in that these authors were limited to possibilities of introducing tensors of rank greater than one. The development of so-called general Riemannian geometries is considered in the second part of this paper. *A. E. Taylor.*

Laugwitz, Detlef. Differentialgeometrie ohne Dimensionsaxiom. II. Riemannsche Geometrie in lokal-linearen Räumen. Math. Z. 61, 134–149 (1954).

This is the second half of a paper published in two installments (see the preceding review, the notation and terminology of which are used here). The author defines what it means for a B-locally linear space to be affinely connected, and shows that for such a space there is an affine connection Γ^i_{hj}, which, in any given coordinate system, is a function with values which are multilinear functionals on $B\times B\times B_1$. The transformation law for the connection involves first and second order Fréchet differentials. The connection is then used to display what occurs in the infinitesimal parallel displacement of a tensor, and to define covariant differentiation. Geodesic lines are characterized by a differential equation involving the affine connection, and an existence theorem is proved: At any point, in any direction from the point, there is a unique geodesic arc. For this the space is assumed to be affinely connected and of class 3 (a differentiability requirement on the coordinate systems). A tensor analogous to the classical Riemann tensor is introduced, and various relations involving it are derived. It is shown that, under suitable conditions, a submanifold of an affinely connected space is affinely connected.

A B-locally linear space is called Riemannian if for each x there is a symmetric bilinear functional $g_{ik}(x)$ on $B\times B$, with certain prescribed properties, which is used to define a differential metric in the space. As a result of the requirements on g_{ik}, it turns out that for each x the topology of B

engendered by the norm $\|\xi^k\|$ is the same as that engendered by taking $\{g_{ik}(x)\xi^i\xi^k\}^{1/2}$ as norm, and that with the latter norm B is a Hilbert space (or Euclidean, if finite-dimensional). A Riemannian space is affinely connected, and there is an explicit formula for $\Gamma^i_{kj}(x)$ in terms of $g_{ik}(x)$. A surface in a Riemannian space is a Riemannian space.

The paper concludes with some detailed results about the theory of curves in infinite-dimensional Riemannian spaces. Generalized Frenet formulas (first deduced for a function-space by Kowalewski) are derived. It is shown that a countable number of curvature functions furnish a complete system of invariants for a curve, even in a space of uncountably many dimensions. *A. E. Taylor* (Mainz).

Viguier, G. Canonisation géométrique de l'équation d'Abel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **41**, 35–50 (3 plates) (1955).

Bernhart, Arthur. Curves of pursuit. Scripta Math. **20** (1954), 125–141 (1955).

Tenca, Luigi. Una particolare elica sferica. Boll. Un. Mat. Ital. (3) **9**, 451–454 (1954).

Dumitrescu, Marin Gh. On ruled surfaces of minimum area. Gaz. Mat. Fiz. Ser. A. **6**, 446–447 (1954). (Romanian)

Muracchini, Luigi. Ancora sulle varietà V_3 analitiche pluririgate. Boll. Un. Mat. Ital. (3) **9**, 262–265 (1954).

Démonstration générale du fait qu'une variété analytique V_3, d'un espace S^n ($n>10$) contenant un système ∞^3 de droites, telles que par chaque point de V_3 il en passe deux, sans que trois droites puissent être coplanaires, est lieu d'une simple infinité de quadriques ordinaires. La démonstration donnée précédemment [cf. Muracchini, même Boll. (3) **8**, 138–144 (1953); MR **15**, 154] se limitait au cas où la V_3 ne satisfaisait pas à d'autres équations que les deux de Laplace exprimant qu'elle est biréglée. Etude des deux cas écartés par cette hypothèse. a) En dehors des équations de Laplace, la V_3 satisfait à des équations du 3ème ordre qui ne s'en déduisent pas. Alors les quatre formes associées du 3ème ordre ne sont plus linéairement indépendantes; l'application du théorème de Cartan établit leur intégrabilité, d'où le résultat. b) La V_3 satisfait à d'autres équations de Laplace qui seront au plus deux et éventuellement à des relations du 3ème ordre; ce cas se ramène au précédent.
B. d'Orgeval (Alger).

Jonas, Hans. Zur Theorie der konjugierten Systeme mit gleichen Invarianten und der Guichard-Calapsoschen *G*-Flächen. Math. Nachr. **12**, 75–112 (1954).

This paper deals with a certain form of the *G*-surfaces introduced by Guichard [C. R. Acad. Sci. Paris **132**, 249–251 (1901)] characterised by a simple form of their Laplace equation. The first four sections have a preliminary character. In the fifth section the differential equation of the surfaces under consideration is established. The two following sections deal with biharmonic transformations of these surfaces and their commutability. The net of lines of curvature generated by two sets of the surfaces and the construction of such a surface from a Laplace transformation are considered in the last section. *J. A. Schouten* (Epe).

Rozet, O., et Legrain-Pissard, N. Sur les congruences *W* dont une des nappes focales est une surface ayant ses asymptotes des deux familles dans des complexes linéaires. Bull. Soc. Roy. Sci. Liège **23**, 280–296 (1954).

Ce travail est consacré à une étude analytique des congruences *W* de droites admettant, pour une de leurs nappes focales, une surface dont les asymptotiques sont situées dans des complexes linéaires. La méthode suivie est basée sur les formules établis par L. Godeaux [La théorie des surfaces et l'espace réglé, Hermann, Paris, 1934]. Après avoir rappelé les généralités indispensables, les auteurs envisagent successivement les trois cas où la nappe focale dont les asymptotiques appartiennent à des complexes linéaires est une surface de Terracini de 1ère, de 2ème ou de 3ème espèce, en séparant les cas où la congruence *W* considérée appartient ou non à un complexe linéaire, et, dans chaque cas, examinent les particularités présentées par la deuxième nappe de la congruence. Cette deuxième nappe est en général une surface à réglées gauches asymptotiques situées dans des complexes linéaires, et sa considération amène les auteurs à la détermination de familles simplement infinies de surfaces admettant les mêmes quadrilatères de Demoulin.
P. Vincensini (Marseille).

Godeaux, Lucien. Sur quatre suites de Laplace associées à une congruence *W*. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 880–885 (1954).

Etant donnée une congruence $W(j)$ et ses deux nappes focales (x), $(\bar{x})$, l'auteur envisage les quatre suites de l'espace à cinq dimensions S_5 qu'il a autrefois associées à (j) [Ann. Soc. Polon. Math. **7**, 213–226 (1929)], à savoir: La suite L associée à (x) contenant les images sur l'hyperquadriques de Klein Q de S_5 des tangentes asymptotiques de (x), la suite analogue $\bar{L}$ relative à $(\bar{x})$, une suite $\mathfrak{T}$ contenant le point J image de j sur Q, et une suite P polaire de $\mathfrak{T}$ par rapport à Q. $\mathfrak{T}$ est inscrite dans L et $\bar{L}$ et P leur est circonscrite. Il envisage le cas où L et $\bar{L}$ s'arrêtent aux points U_n et $\bar{U}_{n+1}$ en présentant le cas de Laplace. Ces suites s'arrêtent en V_{n+2} et V_{n+3} en présentant le cas de Goursat, et $\mathfrak{T}$ s'arrête à $J_{n+1}\equiv\bar{U}_{n+1}$ et à $J_{-(n+3)}\equiv V_{n+3}$. Il montre que P s'arrête à $P_{-(n+1)}\equiv U_n$ et à $P_{n+2}\equiv V_{n+3}$. Tout comme L et $\bar{L}$, les suites $\mathfrak{T}$ et P s'arrêtent par suite dans un sens suivant le cas de Laplace et dans l'autre suivant le cas de Goursat.
P. Vincensini (Marseille).

Godeaux, Lucien. Sur une congruence *W* particulière. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 983–989 (1954).

Une congruence étant représentée sur la quadrique de Klein de l'espace S_5, et $[\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_n, \cdots]$ et $[\cdots, \bar{U}_n, \cdots, \bar{U}_1, \bar{U}, \bar{V}, \cdots, \bar{V}_n, \cdots]$ étant les suites de Laplace associées, l'auteur signale et étudie un type particulier de congruences *W* jouissant de la propriété que les points U_n et $\bar{U}_n$ satisfont aux mêmes équations de Laplace, de même que les points V_n et $\bar{V}_n$. Il montre que la congruence du type envisagé appartiennent à un complexe linéaire.
P. Vincensini (Marseille).

Godeaux, Lucien. Sur une correspondance entre surfaces avec conservation des asymptotiques. Bull. Sci. Math. (2) **78**, 139–146 (1954).

Associated with a point x of a surface (x) in a projective space of three dimensions, there is a known sequence of quadrics $\phi, \phi_1, \cdots, \phi_n, \cdots$, where ϕ is the quadric of Lie. In this sequence any two consecutive quadrics ϕ_n, ϕ_{n+1} touch each other at four points which are characteristics of each of the two quadrics. A necessary and sufficient condition is

obtained for the asymptotic curves of the four common sheets of the envelopes of the quadrics ϕ_n, ϕ_{n+1}, as the point x varies on the surface (x), to correspond to the asymptotic curves of the surface (x). *C. C. Hsiung.*

Manara, Carlo Felice. Invarianti proiettivi differenziali nello spazio e curve *W*. Boll. Un. Mat. Ital. (3) 9, 237–240 (1954).

In an ordinary space, consider three sets of elements (A_i, a_i, α_i), $i=1, 2, 3$, where A_i is a point, a_i a line through A_i and α_i a plane through a_i. Let r_i for each i be the line through A_i and intersecting the two lines a_j $(j\neq i, j=1, 2, 3)$, and let α_{ij} be the plane formed by the two lines a_i and r_j. Certain curves and, particularly, twisted cubics are characterized by means of the three cross-ratios $(\alpha_1, \alpha_{11}, \alpha_{12}, \alpha_{13})$, $(\alpha_2, \alpha_{22}, \alpha_{23}, \alpha_{21})$, $(\alpha_3, \alpha_{33}, \alpha_{31}, \alpha_{32})$ associated with any three points of each curve. *C. C. Hsiung* (Bethlehem, Pa.).

Marcus, F. Sur certaines surfaces R_0. Acad. Repub. Pop. Romåne. Stud. Cerc. Mat. 4, 437–452 (1953). (Romanian. Russian and French summaries)

Let S_x be a surface generated by a point x in a three-dimensional projective space, x_1 and x_{-1} be the points of intersection of a second canonical line and the two asymptotic tangents of S_x at x, and x_i, x_{-i} $(i=2, 3, \cdots)$ be the points defined successively by continuing this process. If x_1x_{-1} is the second edge of Green, then the surfaces S_{x_i} or $S_{x_{-i}}$ $(i=1, 2, \cdots)$ generated by x_i or x_{-i} respectively are projectively applicable to S_x. S_x is also determined such that S_{x_i} or $S_{x_{-i}}$ $(i=1, 2, \cdots)$ are projectively applicable to S_x for the cases where x_1x_{-1} is either a general second canonical line or the second directrix of Wilczynski. *C. C. Hsiung.*

Mihăilescu, Tiberiu. Sur les directrices Wilczynski et les surfaces minima projectives. Acad. Repub. Pop. Romåne. Stud. Cerc. Mat. 1 (1950), 374–392 (1951). (Romanian. Russian and French summaries)

In this paper, the author studies the stratifiability of the two congruences C_1, C_2 respectively formed by the first and second directrices of Wilczynski associated to a proper surface S in a three-dimensional projective space. It is proved that if the two congruences C_1, C_2 form a stratifiable couple, then the surface S belongs to a class of projectively deformable surfaces sustaining an R-net, and this class is identical to a class of surfaces whose asymptotics belong to linear complexes. *C. C. Hsiung* (Bethlehem, Pa.).

Vyčichlo, F. On certain projective invariant planes. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 3, 41–47 (1953). (Czech. Russian summary)

Sind x_1, x_2, x_3, x_4 homogene Koordinaten einer Fläche Φ im projektiven dreidimensionalen Raum und O_1, O_2, O_3, O_4 die Eckpunkte des Koordinatentetraeders, so kann man sich die Aufgabe stellen, projektivinvariant ausgezeichnete Eigenschaften der Fläche Φ in einer Umgebung des Punktes O_4 auf Φ aus dem Verhalten einer algebraischen Näherungsfläche Π abzuleiten. Dabei empfiehlt es sich, die Eckpunkte O_1, O_2 auf den Asymptotentangenten der Fläche durch O_4 zu wählen. Die Approximation von Φ durch Π läuft auf eine Taylorentwicklung um O_4 heraus, die auf die ersten n Glieder beschränkt bleibt. Ist dann α die Tangentialebenen von Π (Φ) im Punkte O_4, so ergibt α mit der kubischen Polarfläche von Π im Punkte O_4 eine kubische Schnittkurve, deren Wendepunkte mit O_4 die Darbouxschen Tangenten der Fläche Π (Φ) im Punkte O_4 bestimmen. Ihre Gleichung lautet $a_1x_1^3+a_2x_2^3=0$. Wird nun Π mit einer Quadrik derart geschnitten, dass die Schnittkurve im Punkt O_4 einen dreifachen Punkt hat, von dessen Tangenten zwei mit der Geraden $x_1-kx_2=0$ zusammenfallen, so genügt die dritte Tangente der Gleichung $k^3a_1x_1+a_2x_2=0$ und die beiden Geraden sind notwendig Darbouxsche Geraden. Derartige quadratische Flächen (Darbouxsche Quadriken) bilden in O_4 ein Bündel, dessen Gleichung angegeben wird. Weiterhin wird der Kegelschnitt bestimmt, der in O_3 mit einer gegebenen algebraischen Kurve eine fünfpunktige Berührung hat. Es folgt die Definition der Moutardschen Quadrik im Punkte O_4 die zur Tangente t gehört. Insbesondere wird die Gleichung der Moutardschen Quadrik in dem Fall bestimmt, dass t durch $\overline{O_3O_4}$ gegeben ist. Moutard's Quadrik der Fläche Π im Punkte O_4 zugehörig zur Tangente t schneidet Π in einer Kurve, die in O_4 einen dreifachen Punkt hat, von dessen Tangenten zwei mit t zusammenfallen. Ist M ein beliebiger Punkt der Tangentialebenen α der Fläche Π im Punkte O_4, so gehört zu M eine Polarebene μ in Bezug auf die zu O_4 und $\overline{O_4M}$ gehörige Moutard'sche Quadrik. Diese Zuordnung heisst die Moutard'sche Korrespondenz. Ersetzt man $\overline{O_4M}$ durch die konjugierte Gerade $\overline{O_4M'}$, so entsteht aus der Moutard'schen Korrespondenz die Segresche Korrespondenz. Im Sonderfall liegen die Punkte der Ebene α, denen in der Moutardschen bzw. Segreschen Korrespondenz Polarebenen der Darbouxschen Quadriken entsprechen, auf den Darbouxschen Tangenten des Punktes O_4. Schliesslich wird die Gleichung der sogenannten Strazzerschen Kurve dritter Ordnung durch O_4 auf Π aufgestellt und die der zweiten "Greenschen Kante", für die auch ein Konstruktionsverfahren gewonnen wird. *M. Pinl* (Köln).

Kovancov, N. I. A spatial indicatrix of geodesic torsions of a triorthogonal system of nonholonomic surfaces. Dokl. Akad. Nauk SSSR (N.S.) 97, 773–776 (1954). (Russian)

If s is the arc length of a curve in a surface and t, n, b are the unit vectors, tangent, surface normal and $b=t\times n$, then the generalized Frenet formulas give $dt/ds=k_nn-k_gb$, $dn/ds=-k_nt+\kappa_gb$, $db/ds=k_gt-\kappa_gn$, where k_n is the normal curvature, k_g the geodesic curvature and κ_g the geodesic torsion. Let $AI_1I_2I_3$ be a moving trihedral, vertex A and I_1, I_2, I_3 unit vectors. Then $dA=\omega_iI_i$, $dI_i=\omega_{ik}I_k$, where the ω's are differential forms with $\omega_{ik}=-\omega_{ki}$. If ω_i are taken as a basis, one can put $\omega_{12}=p_i\omega_i$, $\omega_{23}=q_i\omega_i$, $\omega_{31}=r_i\omega_i$. By means of the Frenet formulas the author shows that $p_1=k_{n_1}$, $q_1=\kappa_{g_1}$, $r_1=k_{g_1}$ for the curve $\omega_1=0$. By cyclic permutations of the subscripts one arrives at the geometrical meaning of the other parameters. The surface orthogonal to I_1, is given by the Pfaffian form $\omega_1=0$; it will be holonomic if the form is completely integrable, in which case $p_3+r_2=0$ and the indicatrix of its torsion is given by $x=0$,

$$r_2y^2+(p_2+r_3)yz+p_3z^2=\pm1,$$

with similar equations for the other orthogonal surfaces. These three conics lie on the surface

$$r_2y^2+p_3z^2+q_1x^2+(r_1+q_2)xy+(p_2+r_3)yz+(q_3+p_1)zx=\pm1,$$

which is the torsion indicatrix for the triorthogonal system. If ω_1, ω_2, ω_3 are all completely integrable,

$$p_3+r_2=q_1+p_3=r_2+q_1=0,$$

which implies that $p_3=r_2=q_1=0$. Thus the above sections of the indicatrix are rectangular hyperbolas.

M. S. Knebelman (Pullman, Wash.).

Oprea, A. Interprétation tangentielle du groupe euclidien. Acad. Repub. Pop. Române. Fil. Iaşi. Stud. Cerc. Şti. 4, 53–67 (1953). (Romanian. Russian and French summaries)

This paper is a continuation of one by A. Haimovici [Acad. Repub. Pop. Române. Bul. Şti. A. 1, 119–124 (1949); MR 14, 1121]. Given a surface S in 3-space, the author considers relations between it and the surface Σ, which is the inverse with respect to the origin O, of the surface ("podaire") whose points are the vertical projections of O on the tangent planes to S. The curvatures K and K_t of S and Σ satisfy $KK_t=\cos^4\theta$, when θ is the angle between OM and the vertical to S at M. The surfaces with $KK_t=$const. are determined, and other similar properties are studied. *A. Nijenhuis* (Princeton, N. J.).

Blanuša, Danilo. Immersion de tores euclidiens à parallélogramme fondamental de forme quelconque dans un espace sphérique ou elliptique à trois dimensions. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 9, 15–25 (1954). (Serbo-Croatian summary)

The Euclidian torus with fundamental rectangle of length a and width b can be imbedded in a 3-sphere $S^3\subset R^4$ of radius $(a^2+b^2)^{1/2}$ in a very natural fashion: $x_1=a\sin 2\pi ua^{-1}$, $x_3=b\sin 2\pi vb^{-1}$, and cos instead of sin for x_2, x_4. It seems convenient to replace $2\pi ua^{-1}$ and $2\pi vb^{-1}$ by $\xi+\eta$ and $\xi-\eta$ respectively. The isometric imbedding of the topological torus $x_1=\varphi(\xi)\sin(\xi+\eta)$, $x_3=\psi(\xi)\sin(\xi-\eta)$, etc., with $\phi^2+\psi^2=1$ in S^3 of radius 1 is obviously also possible. The author proves that this torus, for suitable ϕ and ψ, has any given fundamental parallelogram. His methods are quite elementary and intuitive, and give a considerable amount of additional information on the imbeddings.

A. Nijenhuis (Princeton, N. J.).

Gerretsen, J. C. H. Osservazioni sulla geometria differenziale delle varietà negli iperspazi. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 9, 61–80 (1952).

From the author's summary: "Qui vorrei sviluppare rapidamente, seguendo le idee suesposte. alcuni argomenti di geometria differenziale delle varietà di uno spazio euclideo; ciò intendo fare sulla base dei più elementari concetti del calcolo vettoriale, e cioè l'addizione, la moltiplicazione per un numero ed il cosiddetto prodotto interno." The topics discussed are: the metric tensor, covariant derivative, geodesic curvature and torsion and normal curvature, curvature tensor with its corresponding identities and fundamental equations. *V. Hlavatý* (Bloomington, Ind.).

Pan, T. K. Correction to "On a generalization of the first curvature of a curve in a hypersurface of a Riemannian space." Canad. J. Math. 7, 144 (1955).

See same J. 6, 210–216 (1954); MR 15, 827.

Nash, John. C^1 isometric imbeddings. Ann. of Math. (2) 60, 383–396 (1954).

This paper contains some surprising results on the C^1-isometric imbedding into an Euclidean space of a Riemannian manifold with a positive definite C^0-metric. The theorems are: 1) Any closed Riemannian n-manifold has a C^1-isometric imbedding in E^{2n} (the Euclidean space of dimension $2n$). 2) Any Riemannian n-manifold has a C^1-isometric immersion in E^{2n} and an isometric imbedding in E^{2n+1}. 3) If a closed Riemannian n-manifold has C^1-immersion or imbedding in E^k with $k\geqq n+2$, it also has respectively an isometric immersion or imbedding in E^k. The basic idea is a perturbation process defined in a neighborhood and relative to two normal vector fields. The imbedded or immersed manifold is of course generally quite pathological.

S. Chern (Chicago, Ill.).

Sinyukov, N. S. On geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces. Dokl. Akad. Nauk SSSR (N.S.) 98, 21–23 (1954). (Russian)

If a Riemannian V_n $(n>2)$ allows a non-trivial geodesic mapping on a symmetrical $\bar{V}_n$ (that is, a V_n for which the covariant derivative of the curvature tensor vanishes), then such V_n is of constant curvature. The proof depends on the fact that in such a case the covariant derivative of either the projective curvature tensor or the tensor

$$R^{p}_{\cdot ijk}+\Delta n^{-1}(\delta_k{}^p g_{ij}-\delta_j{}^p g_{ik})$$

vanishes, where $\Delta=\psi_{\alpha\beta}g^{\alpha\beta}$, $\psi_{\alpha\beta}=\partial_\beta\psi_\alpha-\psi_\alpha\psi_\beta$, and the geodesic mapping is given by $\bar{\Gamma}^k{}_{ij}=\Gamma^k{}_{ij}+\delta_i^k\psi_j+\delta_j^k\psi_i$. If a V_n $(n>2)$ does not allow such a mapping, then it is not a V_n of constant curvature. *D. J. Struik* (Cambridge, Mass.).

Takizawa, Seizi. On the characteristic classes of a submanifold. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 241–251 (1954).

The author introduces differential forms in different frame bundles of a Riemannian manifold to study the characteristic classes of the tangent and normal bundles of a submanifold. *S. Chern* (Chicago, Ill.).

Palais, Richard S. A definition of the exterior derivative in terms of Lie derivatives. Proc. Amer. Math. Soc. 5, 902–908 (1954).

This paper starts with a coordinate-free treatment of the Lie derivative belonging to a one-parameter quasi-group of transformations of a manifold of class C^∞. Let $M_1,\cdots,M_{k+1}$ be $k+1$ vector fields and θ a k-form (an alternating covariant tensor field of rank k). The exterior derivative of θ is defined as the $(k+1)$-form Φ such that for any $k+1$ vector fields M_i we have

$$\Phi\langle M_1,\cdots,M_{k+1}\rangle =\tfrac{1}{2}\sum_1^{k+1}(-1)^{i+1}\{M_i[\theta\langle M_1,\cdots,\hat{M}_i,\cdots,M_{k+1}\rangle] +M_i[\theta]\langle M_1,\cdots,\hat{M}_i,\cdots,M_{k+1}\rangle\},$$

$M_i[\;]$ denoting the Lie derivative with respect to the vector field M_i. *J. Haantjes* (Leiden).

Dalla Volta, Vittorio. Sulla geometria differenziale dello spazio delle matrici simmetriche. Ann. Mat. Pura Appl. (4) 37, 291–332 (1954).

In the space of complex matrices $Z=X+iY$ of a finite dimension >1 it has been known that in the half-space $Y>0$ for the metric $ds^2=\mathrm{tr}(Y^{-1}dZY^{-1}d\bar{Z})$ the Riemannian two-dimensional sectional curvature is non-negative and in some positions actually zero. The paper is devoted to the task of determining all such positions in a certain specific form. The description is too elaborate to reproduce, and we only note that it employs manifolds which are determinable by means of invariants attaching to the coefficients of the characteristic equation

$$|Y^{-1}dZY^{-1}d\bar{Z}-\rho I|=0.$$

The author also verifies specifically (what can be inferred from the theory of symmetric spaces in general) that there are no other isometries for the given metric than those result-

ing from the matrix transformations $Z'=(ZC+D)^{-1}(ZA+B)$ which leave the half-space invariant. *S. Bochner.*

∗Geometry of complex domains, a seminar conducted by Professors Oswald Veblen and John von Neumann, 1935–36. Lectures by O. Veblen, and J. W. Givens. Notes by A. H. Taub and J. W. Givens. Rev. ed. The Institute for Advanced Study, Princeton, N. J., 1955. iii+259 pp.

A new edition with minor changes of the mimeographed notes issued in 1936. The whole has been retyped for this edition and reproduced by a method giving much better legibility than the first edition.

Mutō, Yosio. On a curved affinely connected space admitting a group of affine motions of maximum order. Sci. Rep. Yokohama Nat. Univ. Sect. I. 1954, no. 3, 1–12 (1954).

The author proves that the complete group of collineations in an affinely connected space V_n is either of order n^2+n in the case of a flat space, or of order $\leqq n^2$. This theorem and a number of related ones are due to Egorov; it was also proved by Vrănceanu [Acad. Repub. Pop. Romåne. Bul. Şti. A. 1, 813–821 (1949); MR 14, 1123]. The author exhibits a space with an n^2-parameter group and gives the finite equations of this group. The method is purely algebraic but quite elaborate. *M. S. Knebelman.*

Yano, Kentaro, and Hiramatu, Hitosi. On groups of projective collineations in a space of K-spreads. J. Math. Soc. Japan 6, 131–150 (1954).

The authors have shown in an earlier paper [Compositio Math. 10, 286–296 (1952); MR 14, 796] that the projective geometry of K-spreads is equivalent to the theory of a space of elements $(x^i, p_\alpha^{\ i})$ with normal projective connection whose family of K-dimensional totally geodesic subspaces is given by

$$(1)\qquad \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma}+H^i_{\beta\gamma}(x,p)=0\quad \left(p_\alpha^{\ i}=\frac{\partial x^i}{\partial u^\alpha}\right).$$

In that paper they also give the components Π^i_{jk} of the normal projective connection referred to a 'semi-natural' frame of reference.

The authors in this paper consider the conditions under which an infinitesimal point transformation

$$(2)\qquad \bar{x}^i=x^i+\xi^i(x)dt$$

can represent a projective collineation of the K-spreads determined by (1). This condition has been expressed by various authors in terms of the Lie derivative of the projective connection parameters Π^{*i}_{jk} introduced by T. Y. Thomas and J. Douglas. It is pointed out that the study of collineations in terms of the connection parameter Π^i_{jk} is equivalent to using the semi-natural frames of reference used in the earlier paper. The advantages of this approach are pointed out and a number of theorems enunciated.

E. T. Davies (Southampton).

Kanitani, Jôyô. Sur la forme de Darboux généralisée. I. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 225–239 (1954).

Given two arbitrary forms $H_{ij}\omega^i\omega^j$, $H_{ijk}\omega^i\omega^j\omega^k$ ($i, j, k=1, \cdots, n$), we can define an n-dimensional hypersurface V_n and an n-dimensional space R_n having a dominant projective connection without torsion such that V_n has a contact of the fourth order with R_n. In this paper H_{ijk} are determined for given H_{ij} such that V_n has a contact of the sixth order with R_n. *C. C. Hsiung* (Bethlehem, Pa.).

Gotô, Yûzô. On a two-dimensional projectively connected space in the wide sense with torsion. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 253–265 (1954).

In this paper, the author obtains and geometrically interprets three conditions each of which is necessary and sufficient for a two-dimensional space with a dominant projective connection, which can not be imbedded in a three-dimensional projective space, to admit an osculating surface of the seventh order. If a two-dimensional space having a projective connection without torsion admits an osculating surface of the sixth order, then the connection is symmetric.

C. C. Hsiung (Bethlehem, Pa.).

Barthel, Woldemar. Über das Verhältnis der Vektorübertragung zu den Variationsproblemen in Cartanschen Räumen. Rend. Circ. Mat. Palermo (2) 3, 270–281 (1954).

By means of the metric function of a Cartan space [Les espaces métriques fondés sur la notion d'aire, Hermann, Paris, 1933] the area of hypersurfaces as well as the lengths and covariant derivatives of vectors may be defined. The author shows that autoparallel curves do not in general coincide with extremal curves. The differential equations for minimal hypersurfaces is derived, and it is found that hypersurfaces with vanishing second fundamental form do not fall under this category. The author derives necessary conditions which the coefficients of a different connection would have to satisfy in order that extremal properties imply autoparallelism, but no explicit form of these coefficients is given. *H. Rund* (Toronto, Ont.).

Hiramatu, Hitosi. On some properties of groups of homothetic transformations in Riemannian and Finslerian spaces. Tensor (N.S.) 4, 28–39 (1954).

This paper is concerned with further extensions of theorems on groups of motions in a Riemannian space R_N due to K. Yano [Trans. Amer. Math. Soc. 74, 260–279 (1953); MR 14, 688] and in a Finsler space F_N due to H. C. Wang [J. London Math. Soc. 22, 5–9 (1947); MR 9, 206]. The author's treatment is largely based on his Theorem I: If G_r is an r-parameter group of homothetic transformations in R_N, then the maximal subgroup M_{r-1} of motions is a normal subgroup of order $r-1$ and the factor group G_r/M_{r-1} is isomorphic to the multiplicative group L of real numbers. It is asserted that this holds also for F_N. A typical example of the author's results is the concluding theorem: In an F_N with $N>2$, $N\neq4$, there exists no group of homothetic transformations of order r such that

$$\tfrac{1}{2}N(N+1)+1>r>\tfrac{1}{2}N(N-1)+2.$$

H. Rund (Toronto, Ont.).

Moór, Arthur. Ergänzung zu meiner Arbeit: "Über die Dualität von Finslerschen und Cartanschen Räumen." Acta Math. 91, 187–188 (1954).

The following two remarks are given which arose from a result of A. Deicke [Arch. Math. 4, 45–51 (1953); MR 14, 1017]. (1) In the author's paper [Acta Math. 88, 347–370 (1952); MR 14, 689] one must take into consideration only such Finsler and Cartan spaces whose metrics are not positive definite and, accordingly, all investigations must be restricted to a partial domain of the fundamental elements such that in it the metric $L>0$. (2) There are given

examples of Finsler and Cartan spaces with $A_i=0$ which are not Riemannian. These examples can be derived from those of the same property given in footnote 39 on p. 161 of the paper by L. Berwald [Compositio Math. 7, 141–176 (1939); MR 1, 89] by means of a conformal transformation of the form: $\bar{g}_{ik}=\varphi(x)g_{ik}$. Evidently, this transformation holds the relation $A_i=0$ invariant. *A. Kawaguchi.*

Mizoguti, Yukitoyo. Theory of path structure. I. Proc. Japan Acad. 30, 1–8 (1954).

This paper is an exceedingly compact summary of the results, with proofs omitted, of an extensive investigation of the geometry of paths deriving from the original formulation by Veblen and Eisenhart and the algebraic axiomatization of Prenowitz. Most of the present work is said to refer to flat manifolds but extensions to more general cases are envisaged.

A path structure $(S; -)$ is a set S of points with binary combination $-$ such that for each ordered pair of points a, b there is associated a set of points $a-b$. Always extending laws of combination from elements to sets by letting the arguments vary over the sets (e.g., for A, B non-empty sets of points, $A-B=\bigcup\{a-b;\ a \text{ in } A,\ b \text{ in } B\}$), and defining $a+b=\{x;\ a \text{ in } x-b\}$, the axioms are: $a-a=a$; $a+b=b+a$; $a+(a-b)\subset a-b$; if $(a+b)\cap(a+c)\neq 0$ and $b\neq c$, then b is in $a+c$ or c is in $a+b$; and, if $a-b\neq 0$, then $a+b\neq 0$. If $a=b$ or $a\neq b$, $a+b\neq 0$, x and $y\neq x$ in $a+b$ implies a in $x-y$ or $y-x$, then $a+b$ is an open segment and $a\oplus b=a\cup(a+b)\cup b$ is closed. After suitable definitions, a path is defined as the set union of all the segments of a maximal connected collection of segments. If, for $a-b\neq 0$ and $b-a\neq 0$, $(a+b)-x$ remains the same for all x in $a+b$, $(S; -)$ is called a linear structure and the path generated by $a+b$ is

$$a\vee b=a\cup b\cup(a+b)\cup(a-(a+b))\cup(b-(a+b)).$$

A theory of convex sets is next briefly indicated. The structure of the convex hull of a finite or infinite set of points is said to be obtained in detail by a new method (no ambient affine space being assumed). Linear structures satisfying various associative laws are now characterized: if $+$ is associative, each component (=maximal subtractively arcwise connected set) is a descriptive geometry; if $\oplus$ is associative, a component is a single line, a descriptive geometry or a spherical geometry; if ∇ is associative for $a\nabla b=(a+b)\cup(a-(a+b))\cup(b-(a+b))$, components (no longer requiring "subtractively") are projective geometries; if $\vee$ is associative and $a-b\neq 0$ implies $b-(a+b)\neq 0$, then a component is a spherical or projective geometry.

The last three pages consider such topics as region systems, path manifolds and homotopy (if a path manifold "has a universal homotopy, then it has a connected and simply connected covering manifold"). Many other results are stated in the paper. *J. W. Givens* (Princeton, N. J.).

Mihăileanu, N. N. Objets géométriques en géométrie différentielle. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 1 (1950), 318–373 (1951). (Romanian. Russian and French summaries)

Let ω and Ω be geometrical objects in X_1 with the following transformations (under the transformation $\tau=\tau(t)$ of the variable t)

$$\omega=\tau'\bar{\omega}+\tau''(\tau')^{-1};\quad \Omega=\tau'^2\bar{\Omega}+\{\tau\},$$

$\{\tau\}$ being the Schwarzian derivative of τ. Then from invariants s of weight p $(s=(\tau')^p\bar{s})$ by an invariant process new invariants $\mathcal{J}$ and $\mathcal{K}$ can be obtained:

$$\mathcal{J}=s'-p\omega s,\quad \mathcal{K}=ss''-(1+\tfrac{1}{p}p^{-1})s'^2-p\Omega s^2.$$

This method is applied to obtain invariants determined by a linear differential equation. As a generalization the author considers geometrical objects depending on two variables u and v under the transformations $u=u(\bar{u})$, $v=v(\bar{v})$. Applied to the system of differential equations, which define a surface $x(u, v)$ on its asymptotic lines, projective and affine invariants of the surface are obtained. *J. Haantjes.*

Neumann, Maria. Les objets géométriques associés aux surfaces réglées. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. 2, 445–462 (1951). (Romanian. Russian and French summaries)

The projective invariants of a ruled surface

$$X^h=y^h(x)+uz^h(x)$$

are the invariants of a system of differential equations [expressing y'' and z'' in y', z', y and z; Wilczynski, Projective differential geometry of curves and ruled surfaces, Teubner, Leipzig, 1906] under the transformations $\bar{y}=\alpha(x)y+\beta(x)z$, $\bar{z}=\gamma(x)y+\delta(x)z$ and $\xi=\xi(x)$. It is shown that the theory of geometrical objects by Gołąb [Ann. Soc. Polon. Math. 19, 7–35 (1947); MR 9, 206], Gheorghiu [C. R. Acad. Sci. Paris 227, 613–615 (1948); MR 10, 267] and Mihăileanu [see the paper reviewed above] in an X_1 leads immediately to the invariants found by Wilczynski. A similar method is applied to the differential equations of a ruled surface in the form given by Fubini-Čech. *J. Haantjes* (Leiden).

Renaudie, Josette. Un théorème sur les espaces harmoniques. C. R. Acad. Sci. Paris 238, 199–201 (1954).

Let V be a Riemannian manifold of class C^∞. The following theorem is established: If, for each point $x_0 \in V$, the group of local isometries leaving x_0 invariant is transitive with respect to the directions issuing from x_0, the variety is completely harmonic (i.e., harmonic at each point). *D. C. Spencer* (Paris).

Cartan, Henri, et Serre, Jean-Pierre. Un théorème de finitude concernant les variétés analytiques compactes. C. R. Acad. Sci. Paris 237, 128–130 (1953).

In this note the authors show that if X is a compact complex analytic manifold and if F is a coherent sheaf on X, then the dimension of $H^p(X; F)$ is finite for all p. This was also demonstrated by Kodaira [Proc. Nat. Acad. Sci. U. S. A. 39, 865–868 (1953); MR 16, 74]) who based his proof on the properties of solutions of a system of strongly elliptic differential equations.

Let U be a finite open covering of X. If s^p is a p-simplex of the covering U, then let $C(s^p)$ denote the carrier of s^p. The authors' proof uses the fact that there is a finite open covering of the manifold, call it U, with the property that for each simplex s^p of the nerve of U, $H^p(C(s^p); F)$ is zero for positive p. This allows one to conclude that $H^p(C(s^p); F)$ is zero for positive p. This allows one to conclude that $H^p(N(U); F)$ is isomorphic to $H^p(X; F)$, where $H^p(N(U); F)$ denotes the cohomology of the nerve of U, $N(U)$, with coefficients in the sheaf F.

It is shown that if U_1 is a proper refinement of the covering U, then the homomorphism

$$r:\ Z^p(N(U_1); F)\to Z^p(N(U); F)$$

is completely continuous when the two linear spaces are suitably topologized as Fréchet spaces. Then if u and v are linear mappings of $C^{p-1}(N(U); F)\times Z^p(N(U_1); F)$ into

$Z^p(N(U); F)$ given by (δ, r) and $(0, -r)$ respectively, then u is an onto-mapping and v is completely continuous so that the image of $u+v=(\delta, 0)$ has a finite codimension by a result of L. Schwartz [C. R. Acad. Sci. Paris **236**, 2472–2473 (1953); MR **15**, 233]. Thus $H^p(N(U); F)$ is finite-dimensional and so is $H^p(X; F)$. *P. E. Conner.*

Wang, Hsien-Chung. Closed manifolds with homogeneous complex structure. Amer. J. Math. **76**, 1–32 (1954).

Eine komplexe Mannigfaltigkeit heisst homogen, wenn sie eine transitive Gruppe komplex-analytischer Homöomorphismen auf sich besitzt. In der vorliegenden Arbeit werden die einfach-zusammenhängenden kompakten homogenen komplexen Mannigfaltigkeiten (kurz "C-Räume" genannt) ausführlich untersucht und in gewissem Sinne vollständig aufgezählt. Es ergibt sich eine grosse Zahl neuer Beispiele komplexer Mannigfaltigkeiten, insbesondere solcher, die keine Kähler-Metrik zulassen, also nicht algebraisch sind; sie enthalten die Beispiele von Calabi und Eckmann [Ann. of Math. (2) **58**, 494–500 (1953); MR **15**, 244] als Spezialfälle. Die wichtigsten Ergebnisse der Arbeit: 1) Jeder C-Raum ist homöomorph dem Faktorraum G/H einer kompakten halbeinfachen Lieschen Gruppe G nach einer zusammenhängenden abgeschlossenen Untergruppe H, deren halbeinfacher Bestandteil mit demjenigen des Zentralisators eines Toroids in G zusammenfällt (eine solche Untergruppe H wird kurz "C-Untergruppe" genannt). 2) Umgekehrt besitzt der Faktorraum G/H einer kompakten einfach-zusammenhängenden halbeinfachen Lieschen Gruppe G nach einer C-Untergruppe stets eine homogene komplexe Struktur (falls seine Dimension gerade ist). 3) Ist G wie in 2), aber einfach, und H eine halbeinfache C-Untergruppe von G, so heisst G/H ein "M-Raum" (es werden alle M-Räume aufgezählt). Jede Mannigfaltigkeit M gerader Dimension, die einem Produkt von M-Räumen homöomorph ist, besitzt homogene komplexe Strukturen, und zwar überabzählbar viele nicht-aequivalente; sie sind alle nicht-kählersch, da die 2te Bettische Zahl $=0$ ist. 4) Jeder C-Raum ist Basisraum einer Faserung eines Produktes von M-Räumen in Torusfasern. 5) Ist die 2te Bettische Zahl eines C-Raumes $=0$, dann ist auch seine Eulersche Charakteristik $\chi=0$. 6) Eine einfach-zusammenhängende Mannigfaltigkeit mit $\chi\neq0$ besitzt höchstens endlich viele inaequivalente homogene komplexe Strukturen; mit $\chi=0$ entweder keine oder überabzählbar viele. 7) Die Identitätskomponente der Gruppe aller komplex-analytischen Homöomorphismen eines C-Raumes ist lokal das direkte Produkt einer komplexen halbeinfachen Lieschen Gruppe und einer komplexen Vektorgruppe.

Den Ausgangspunkt bildet die Bemerkung, dass jeder C-Raum als Faktorraum einer komplexen Lieschen Gruppe nach einer komplex-analytischen Untergruppe dargestellt werden kann; dies folgt leicht aus einem Satz von Bochner und Montgomery [Ann. of Math. (2) **48**, 659–669 (1947); MR **9**, 174]. Dadurch wird die Untersuchung im Wesentlichen zu einem Problem über (reelle und komplexe) Liesche Algebren, welches im Zusammenhang mit wichtigen Struktureigenschaften eingehend behandelt wird. *B. Eckmann* (Zürich).

Have

★Eckmann, Beno. Structures complexes et transformations infinitésimales. Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 176–184. Edizioni Cremonese, Roma, 1954. 4000 Lire.

This is a lecture on results obtained in collaboration with A. Frölicher; a detailed account will be given elsewhere. An almost complex structure J on a differentiable manifold V_{2n} (endomorphism of the tangent space $T(V)$ of square -1) is made to operate on the module $M(V)$ of vector-fields in V. By calculation with Poisson brackets $[a, b]$ and suitable affine connections it is shown that the torsion t of J, introduced by the author and Frölicher [C. R. Acad. Sci. Paris **232**, 2284–2286 (1951); MR **13**, 75], the "obstruction against integrability", is given by

$$-4t(a, b)=[a, b]+J[Ja, b]+J[a, Jb]-[Ja, Jb].$$

Other forms for the integrability condition are given, in particular one that says that the eigenspaces of the complexification of $M(V)$ under J must be closed under $[a, b]$. A group operating on V leaves a complex structure invariant if it leaves the induced almost complex structure invariant. A criterion for the existence of complex structures on Lie groups is developed: the complexified Lie algebra must be the direct sum of two conjugate subspaces, which are subalgebras. Applying this criterion and making use of a Weyl basis it is shown that every compact even-dimensional Lie group possesses left-invariant complex structures [for this result cf. also the paper reviewed above and the reviewer, Portugal. Math. **12**, 129–132 (1953); MR **15**, 505]. The criterion is extended to homogeneous spaces G/H: The two conjugate subalgebras appearing in it must now have for intersection the complexified Lie algebra of H. It is stated that this leads to the complete results of Wang [loc. cit]. on complex-homogeneous spaces. *H. Samelson.*

Hirzebruch, Friedrich. Some problems on differentiable and complex manifolds. Ann. of Math. (2) **60**, 213–236 (1954).

The paper lists a set of 34 unsolved problems of current interest concerned with differentiable, almost complex, and complex manifolds. The problems are expertly chosen; their clarification and partial or complete solutions will probably take many years and will certainly mean progress of the field. In stating the problems, the author tries to give their motivation, their relations to known results, and related facts. In this sense the paper is at the same time a resumé and exposition of the subject, or at least of a major part of it.

The following errata were communicated by the author: 1) p. 220, line 4 of Problem 8, read "$10\leq m^*\leq14$" instead of "$10\leq m^*\leq$"; 2) p. 232, line 3 of formula (13), read "P_2 (bigenus) $=\dim H^0(V_2, 2K)=10$"; 3) p. 232, Problem 30, sentence between parentheses should read: (Since because of the vanishing of g_1 and g_2 the linear genus $p^{(1)}$ is just the negative of the second Betti number increased by 11, we can also ask: What is the largest linear genus V_2 can have?). *S. Chern* (Chicago, Ill.).

Have

★Schwartz, Laurent. Courant associé à une forme différentielle méromorphe sur une variété analytique complexe. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 185–195. Centre National de la Recherche Scientifique, Paris, 1953.

Let V be a complex-analytic variety of complex dimension n. A differential form on V is defined to be semi-meromorphic if, in the neighborhood of each point of V, it is the quotient of a differential form of class C^∞ by a holomorphic function. The poles of a semi-meromorphic form are contained in an analytic subvariety W of V if, in the neighborhood of each point, the form is the quotient of a form of class C^∞ by a holomorphic function having zeros only in W. A semi-meromorphic form is called "regular" if, in the neighbor-

hood of each point, it is a finite sum of semi-meromorphic forms each of which has its poles in a non-singular analytic subvariety.

A differential operator D on the space of courants T of V is said to be semi-holomorphic if each point of V has a neighborhood covered by local analytic coordinates $z_1, z_2, \cdots, z_n$ in which each coefficient of DT is a finite linear combination, with coefficients which are functions of class C^∞, of derivatives $(\partial/\partial z_1)^{p_1}(\partial/\partial z_2)^{p_2}\cdots(\partial/\partial z_n)^{p_n}$ of the coefficients of T.

The following theorem is established: There is one and only one operation vp on V which assigns to every regular semi-meromorphic form ω of type (p, q) a courant $vp(\omega)$ of type (p, q) having the following properties: 1) If, locally, ω has coefficients which are summable, then $vp(\omega)=\omega$; 2) the operation vp is linear for the structures of modules over the ring of semi-holomorphic differential operators. An application of this theorem is made to the additive Cousin problem. *D. C. Spencer* (Paris).

Blanchard, André. Espaces fibrés kählériens compacts. C. R. Acad. Sci. Paris **238**, 2281–2283 (1954).

The author establishes the following theorem. Let E be a compact complex-analytic fibre bundle with base B and fibre F. Suppose that the fundamental group of B operates trivially on the first cohomology group with real coefficients of F. Then E is a Kähler variety if and only if the following three conditions are satisfied. 1. The fundamental group of B respects a real cohomology group of dimension 2 of F corresponding to a Kähler metric of F. 2. B has a Kähler metric. 3. The image of $H^1(F, R)$ in $H^3(B, R)$ is zero. *D. C. Spencer* (Paris).

Lichnerowicz, André. Sur les espaces homogènes kählériens. C. R. Acad. Sci. Paris **237**, 695–697 (1953).

Let W be a manifold of dimension $2n$ provided with the structure of a homogeneous space G/H, where G is connected and H is compact. Suppose that W is complex-analytic with a Kähler metric which is invariant under G. The following theorems are given. 1) If G has a non-discrete center then, in terms of a suitable local analytic coordinate system $z^1, z^2, \cdots, z^n$, the metric can be written in the form

$$ds^2=dz^1d\bar z^1+\sum_{\alpha,\beta=2}^{n} g_{\alpha\bar\beta}dz^\alpha d\bar z^\beta,$$

where $g_{\alpha\bar\beta}$ is independent of z^1 and $\bar z^1=\overline{z^1}$. 2) If W is not locally unitary and if G is compact, the connected group of isotropy has a non-discrete center. 3) Suppose that G is compact and semi-simple, H connected, and let Z be the connected component of the identity of the center of H. Then H is the connected component of the identity of the centralizer of Z in G and, in particular, H is a subgroup of G of maximum rank. 4) Under the same hypotheses as in 3) concerning G and H, W is a topological and Riemannian product of homogeneous Kähler spaces $W_i\approx G_i/H_i$, where G_i is simple with center reduced to the identity and where H_i is the connected centralizer of the connected component of the identity of its proper center. *D. C. Spencer*.

***Lichnerowicz, André. Espaces homogènes kählériens.** Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 171–184. Centre National de la Recherche Scientifique, Paris, 1953.

Let V be a differentiable manifold of dimension $2n$ and of class C^3 which possesses an almost-complex structure defined by an operator J on the vectors whose action is described in terms of a tensor F_i^j of class C^2. Let V be assigned an Hermitian structure subordinate to the given almost-complex structure and introduce local linear forms Θ^α, $\Theta^{\bar\alpha}=\bar\Theta^\alpha$ $(\alpha=1, 2, \cdots, n)$ such that $ds^2=2\sum\Theta^\alpha\Theta^{\bar\alpha}$, $F=i\sum\Theta^\alpha\wedge\Theta^{\bar\alpha}$, where F is the exterior quadratic form of rank $2n$ whose coefficients are derived from the tensor F_i^j by means of the Hermitian metric. The frames $(\vec e_i)$ dual to the coframes defined by the (Θ^i) are said to be adapted to the Hermitian structure, the transformation from one adapted frame to another being effected by unitary matrices.

If the Riemannian connection associated with ds^2 is referred to adapted frames, the connection is defined by a system of forms π_i^j whose compatibility with the metric is expressed by the equations $\pi_\beta^\alpha+\pi_{\bar\alpha}^{\bar\beta}=0$, $\pi_{\bar\beta}^\alpha+\pi_{\bar\alpha}^\beta=0$, $\pi_\beta^{\bar\alpha}+\pi_\alpha^{\bar\beta}=0$. Under a change of adapted frame, the transformations of these connection forms are:

$$\pi_{\lambda'}^{\mu'}=A_{\lambda'}^{\alpha}A_\beta^{\mu'}\pi_\alpha^\beta+A_\alpha^{\mu'}dA_{\lambda'}^{\alpha}; \tag{1}$$

$$\pi_{\bar\lambda'}^{\mu'}=A_{\bar\lambda'}^{\bar\alpha}A_\beta^{\mu'}\pi_{\bar\alpha}^\beta, \tag{2}$$

where $(A_{\lambda'}^{\alpha})$ is a unitary matrix. The formulas involving the curvature Ω may be written:

$$d\pi_\alpha^\beta=\pi_\alpha^\rho\wedge\pi_\rho^\beta+\pi_\alpha^{\bar\rho}\wedge\pi_{\bar\rho}^\beta-\Omega_\alpha^\beta; \tag{1'}$$

$$d\pi_{\bar\alpha}^\beta=\pi_{\bar\alpha}^\rho\wedge\pi_\rho^\beta+\pi_{\bar\alpha}^{\bar\rho}\wedge\pi_{\bar\rho}^\beta-\Omega_{\bar\alpha}^\beta. \tag{2'}$$

Let an adapted frame be attached to each point of a neighborhood of V and write $\pi_i^j=\Gamma_{ik}^j\Theta^k$. If ∇_k denotes the covariant derivative with respect to the Riemannian connection, $\nabla_k F_{\alpha\bar\beta}=i\nabla_k g_{\alpha\bar\beta}=0$ (where $g_{\alpha\bar\beta}$ denote the components of the Hermitian metric tensor); moreover, $\nabla_\gamma F_{\bar\alpha\bar\beta}=2i\Gamma_{\bar\beta\gamma}^{\alpha}$, $\nabla_{\bar\gamma}F_{\bar\alpha\bar\beta}=2i\Gamma_{\bar\beta\bar\gamma}^{\alpha}$. If F is closed (i.e., if V is symplectic), $\Gamma_{\bar\beta\gamma}^{\alpha}=0$. If the almost-complex structure of V defined by F_i^j is complex-analytic, $\nabla_{\bar\gamma}F_{\bar\alpha\bar\beta}=2i\Gamma_{\bar\beta\bar\gamma}^{\alpha}$ is symmetric in $\bar\beta$ and $\bar\gamma$ and therefore zero. Finally, the variety V is pseudo-Kählerian if and only if $\pi_{\bar\beta}^{\alpha}=0$.

Suppose that V is complex-analytic. By (1) the forms π define an intrinsic connection $(\pi_\alpha^\beta, \pi_{\bar\alpha}^{\bar\beta})$ in the fibre bundle of adapted frames which is called the Hermitian connection induced by the Riemannian connection. If $\hat\nabla_k$ is the covariant derivative for this connection, $\hat\nabla_k g_{ij}=0$, $\hat\nabla_k F_{ij}=0$, and the fundamental formulas for the connection may be written

$$d\Theta^\alpha=\Theta^\beta\wedge\pi_\beta^\alpha+\hat\Omega^\alpha,\quad d\pi_\alpha^\beta=\pi_\alpha^\rho\wedge\pi_\rho^\beta-\hat\Omega_\alpha^\beta,$$

where $\hat\Omega^\alpha$ (the torsion) and $\hat\Omega_\alpha^\beta$ (the curvature) are given by $\hat\Omega^\alpha=\Gamma_{\bar\beta\gamma}^{\alpha}\Theta^{\bar\beta}\wedge\Theta^\gamma$, $\hat\Omega_\alpha^\beta=\Omega_\alpha^\beta-\pi_\alpha^{\bar\rho}\wedge\pi_{\bar\rho}^\beta$. The variety V is Kählerian if and only if the torsion $\hat\Omega^\alpha$ vanishes.

Let x be a point of V and denote by Φ_x, Ψ_x, ρ_x, σ_x, respectively, the holonomy group, homogeneous holonomy group, restricted holonomy group, and restricted homogeneous holonomy group at x with respect to the Riemannian connection. The corresponding groups at x defined with respect to the connection $(\pi_\alpha^\beta, \pi_{\bar\alpha}^{\bar\beta})$ will be denoted by $\hat\Phi_x$, $\hat\Psi_x$, $\hat\rho_x$, $\hat\sigma_x$.

The following theorems are established. (1) A Riemannian variety V is pseudo-Kählerian if and only if Ψ_x is a subgroup of the real representation of $U(n)$. (2) A pseudo-Kählerian variety has a holonomy group σ_x which is a subgroup of the real representation of $SU(n)$ if and only if its Ricci curvature vanishes. (3) The holonomy group σ_x of a pseudo-Kählerian variety with Ricci curvature different from zero has a non-discrete center. (4) If $V=G/H$ is a homogeneous Riemannian space with Ricci curvature different from zero and with σ_x irreducible, the connected linear isotropy group $\tilde H$ (connected component of the identity of the linear isot-

ropy group) is contained in σ_x. (5) Every homogeneous space G/H with G compact and with Ricci curvature different from zero is locally euclidean. (6) If a homogeneous Kähler space $V=G/H$ of real dimension $2n$ ($n>1$) has a non-zero Ricci curvature and if $\tilde{H}$ is irreducible in the real sense, then V is Hermitian symmetric and irreducible.

The final section of the paper concerns the case where G is compact. *D. C. Spencer* (Paris).

Lichnerowicz, André. Un théorème sur les espaces homogènes complexes. Arch. Math. 5, 207–215 (1954).

In the paper reviewed above the author has shown that every homogeneous Kähler manifold G/H with non-vanishing Ricci curvature, and with the connected component of the identity of the linear isotropy group irreducible (in the real sense), is an Hermitian symmetric space. If the homogeneous space is compact, the vanishing of the Ricci curvature implies by a result of S. Bochner that the space is locally euclidean. In the present note the author indicates how these results can be extended to homogeneous complex spaces which are not necessarily Kählerian. For the notation used here see the preceding review.

Suppose that V is complex-analytic, and let a Hermitian structure be assigned to V. The real quadratic form $\psi=i\Omega_\alpha{}^\alpha=-d(i\pi_\alpha{}^\alpha)$, which is defined on V, is closed and has an inverse image in the bundle of adapted frames which is homologous to zero. It is shown that the group θ_x is a subgroup of $SU(n)$ if and only if $\psi=0$.

Now suppose that $V=G/H$ and that V has a complex-analytic structure which is invariant under G. An Hermitian metric can be assigned to V which is invariant under G, and there exist frames which are adapted simultaneously to the Hermitian structure and to G. Let $\tilde{H}$ denote the connected component of the identity of the linear isotropy group; $\tilde{H}$ can be identified with a subgroup of the real representation of $U(n)$. Suppose that $\tilde{H}$ is irreducible and that the complex dimension n of V is greater than 1. The following two theorems are proved. (1) If $\tilde{H}$ does not belong to $SU(n)$, V is Hermitian symmetric and irreducible. (2) If θ_x belongs to $SU(n)$, the divergence of the transform by J of each Killing vector of G is constant on the space.

D. C. Spencer (Paris).

Legrand, Gilles. Connexions définies sur une variété presque hermitique. C. R. Acad. Sci. Paris 237, 1626–1627 (1953).

An almost Hermitian space is characterized by two tensors $g_{ih}(=g_{hi})$ and $F_{ih}=(-F_{hi})$ of the maximum rank satisfying $g^{ak}F_{ai}F_{kh}=g_{ih}$. The author studies two connexions

(1) $$L^h{}_{ji}=\{{}^{\,h}_{ji}\}-\tfrac{1}{2}(\mathring{\nabla}_jF_{il})F^{lh},$$

(2) $$C^h{}_{ji}=\{{}^{\,h}_{ji}\}-\tfrac{1}{2}(\mathring{\nabla}_jF_{il}+\mathring{\nabla}_iF_{jl}-\mathring{\nabla}_lF_{ij})F^{lh},$$

where $\mathring{\nabla}_j$ denotes the covariant differentiation with respect to the Christoffel symbols $\{{}^{\,h}_{ji}\}$.

The connexion L satisfies always $\nabla_jg_{ih}=0$, $\nabla_jF_{ih}=0$ and reduces to

$$L^\kappa{}_{\mu\lambda}=\{{}^{\,\kappa}_{\mu\lambda}\},\quad L^{\bar\kappa}{}_{\bar\mu\bar\lambda}=\{{}^{\,\bar\kappa}_{\bar\mu\bar\lambda}\}\text{ conj.}$$

the other L's being zero, for a Hermitian space. This is the connexion introduced by A. Lichnerowicz in the paper reviewed above. The connexion C satisfies only $\nabla_jF_{ih}=0$. But in a pseudo-Hermitian space, the connexion C satisfies $\nabla_jg_{ih}=0$ and $\nabla_jF_{ih}=0$. This reduces to

$$C^\kappa{}_{\mu\lambda}=g^{\kappa\bar\nu}\partial_\mu g_{\lambda\bar\nu}\text{ conj.}$$

the other C's being zero, for a Hermitian space. This is the connexion introduced by J. A. Schouten and D. van Dantzig [Math. Ann. 103, 319–346 (1930)] and recently used by S. S. Chern [Ann. of Math. (2) 47, 85–121 (1946); MR 7, 470] and P. Libermann [see the following review].

K. Yano (Amsterdam).

Libermann, Paulette. Sur le problème d'équivalence de certaines structures infinitésimales. Ann. Mat. Pura Appl. (4) 36, 27–120 (1954).

This memoir gives, in its first part, an exposition of Cartan's theory of (local) equivalence, and, in its second part, a detailed study of a number of special cases. A number of the results of the second part have been presented before [Libermann, Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 742–755 (1950); C. R. Acad. Sci. Paris 233, 17–19, 1571–1573 (1951); 234, 395–397, 1030–1032, 2517–2519 (1952); Colloque de Topologie de Strasbourg, 1951, no. VIII, Univ. de Strasbourg, 1952; MR 12, 749; 13, 75, 780; 14, 88; Ehresmann and Libermann, C. R. Acad. Sci. Paris 232, 1281–1283 (1951); MR 12, 749]. Ch. I: Differential systems, using Ehresmann's notion of jet (the q-jet of a function at a point consists of the values of all partial derivatives of order $\leq q$ at the point), Pfaffian systems in involution, Lie pseudo-groups of transformations, including Cartan's "infinite" groups, and their structure equations. Cartan's two fundamental theorems are proved. Ch. II: Cartan's problem of equivalence is viewed as equivalence of infinitesimal regular structures of first order in the sense of Ehresmann, i.e., bundles obtained from the tangent bundle of a manifold by reductions of the structure group; only local equivalence is considered (throughout the paper). Tensorial differential forms, affine connections, covariant differentiation, torsion and curvature are introduced. Integrability of the structure means zero curvature and torsion for a suitable connection. Another theorem characterizes the pseudo-group of local automorphism in the case where the choice of the torsion determines the connection. The equivalence problem is first treated for the restricted case, group = identity, in which case the structure equations lead to the necessary invariants. The general case is reduced to the restricted one by successively constructing subbundles on which as many components of curvature and torsion as possible are constant. The question whether the torsion determines the connection plays a role here. Ch. III is an algebraic study of the various groups of linear transformations on E^n: complex, paracomplex, unitary, quaternionic of first and second kind, etc. A special section concerns quadratic exterior forms Ω of maximal rank, leading to the Hodge-Eckmann-Guggenheimer theorem on effective forms (which is shown to be equivalent to Lepage's theorem). The operator Λ, usually considered here, is shown to depend on Ω only, and not on the associated metric (= symmetric quadratic form). Ch. IV: Symplectic manifolds (i.e. a closed 2-form Ω of maximal rank everywhere is given), in particular the notion of codifferential, harmonicity with respect to Ω [cf. also Guggenheimer, Colloque de Topologie de Strasbourg, Bibliothèque Nat. Univ. de Strasbourg, 1951, no. I], almost-symplectic spaces, torsion form and conformal torsion form [cf. also H. C. Lee, Amer. J. Math. 65, 433–438 (1943); MR 5, 15], integrating factor for Ω. New results concern the torsion associated with an almost symplectic structure. Ch. V: Almost complex (almost hermitian) and almost paracomplex (almost parahermitian) structures; structure equations, integrability conditions for a structure to be true-complex, uniqueness of associated affine connections, isotropic and locally homo-

geneous structures, curvature and torsion, the equivalence problem for 4-dimensional almost complex (and hermitian) structures, the special structures on the sphere S_6 and the quadric Q_6 associated with the exceptional group G_2. Ch. VI is a detailed study of quaternionic structures of the second kind, from the same point of view: affine connections, curvature and torsion, etc. *H. Samelson* (Ann Arbor, Mich.).

have

★Libermann, Paulette. Sur certaines structures infinitésimales régulières. Géométrie différentielle. Colloques Internationaux du Centre National de la Recherche Scientifique, Strasbourg, 1953, pp. 161–170. Centre National de la Recherche Scientifique, Paris, 1953.

Brief exposition of matters which, in part, have been described by the author at other places [see the preceding review]. An infinitesimal regular structure on a differentiable manifold (Ehresmann), is any bundle subordinate to the principal bundle of frames (structure group reduced from $GL(n)$ to some subgroup G). Infinitesimal and affine connections can be constructed [Ehresmann, Colloque de Topologie, Bruxelles, 1950, pp. 29–55, Thone, Liège; MR 13, 159], and, by way of structure equations, torsion and curvature tensor are defined. This is discussed in more detail for several specific groups G [complex group $GL(m, C)$ for $n=2m$, paracomplex group, unitary group, para-unitary group, quaternion group of second kind]; in these cases one can specify certain canonical connections. Isotropic and locally homogeneous structures, conformal torsion and curvature tensors are considered briefly. *H. Samelson.*

have

★Libermann, Paulette. Sur la courbure et la torsion de certaines structures infinitésimales. Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 234–246. Edizioni Cremonese, Roma, 1954. 4000 Lire.

For an infinitesimal regular structure, i.e., for a fiber bundle, with group G, subordinated to the tangent bundle of a manifold, the author describes briefly the concepts of tensorial differential forms, affine connection (the group used here is the extension of G by the translation of n-space), structure equations, curvature and torsion forms, and the characteristic classes, here called Chern forms (formed like the Pontryagin classes). These concepts are then worked out in detail for several specific structures: almost hermitian and parahermitian, almost quaternion of the second kind. Several theorems connecting isotropy of the structures with other properties are given. The last structure has several affine connections; explicit calculation verifies the theorem of Weil that the characteristic forms are independent of the choice of connection, up to derived forms.

H. Samelson (Ann Arbor, Mich.).

have

★Dedecker, Paul. Systèmes différentiels extérieurs, invariants intégraux et suites spectrales. Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 247–262. Edizioni Cremonese, Roma, 1954. 4000 Lire.

The author describes how spectral sequences and their generalizations enter into the theory of exterior differential systems and of integral invariants. Regular homology theory in a differentiable manifold V is defined: the generators of the chains are regular cubes, i.e., regular (non-zero Jacobian) maps of some neighborhood of the cube I^n (in n-space E^n) into V. Sections of cubes are defined as usual; a fragment of a cube is obtained from a parallelepiped in the cube. A subgroup R of the group $C(V)$ of chains is saturated if it has a base formed by cubes and is closed under the formation of sections and fragments. An example is given by the integral chains of an exterior differential system. The basic idea is now that such an R defines a filtration of $C(V)$: An n-cube has filtration $\leq p$, if all vertical (i.e., the first p coordinates fixed) $(n-p)$-sections belong to R. This is a generalization of the filtration studied by Serre in the theory of fiber spaces [Ann. of Math. (2) **54**, 425–505 (1951); MR 13, 574] and gives rise to a spectral sequence. There is a dual development for differential forms. Generalizations of the concept of integral invariant are then introduced: differential forms which annihilate or are constant on certain of the groups occurring in the construction of the spectral sequence [or the more general groups considered by Deheuvels, C. R. Acad. Sci. Paris **235**, 778–780 (1952); MR 14, 492]. The author is led to constructing from the filtration a double sequence of bigraded groups $E_{r,s}$ and differentials $\partial_{r,s}$ of degree $(-r, r-1)$, such that the homology of $E_{r,s}$ is $E_{r+s,s}$; for $s=1$ this is the usual spectral sequence. An application states that (roughly speaking) the set of extremal cubes of an exterior variational problem is characterized by the fact that it admits an integral invariant of a certain type, generalizing E. Cartan's classical result concerning simple integrals; as the author has shown [Géometrie différentielles, Colloq. Internat. Centre. Nat. Rech. Sci., Strasbourg, 1953, pp. 17–34; MR 16, 50], every first order variational problem is equivalent to an exterior one.

H. Samelson (Ann Arbor, Mich.).

Ōtsuki, Tominosuke, and Tashiro, Yoshihiro. On curves in Kaehlerian spaces. Math. J. Okayama Univ. **4**, 57–78 (1954).

For a real parameter t, the authors say that a curve $s^\lambda=s^\lambda(t)$ in an Hermitian metric is holomorphically flat if it satisfies the equation

$$\frac{d^2s^\lambda}{dt^2}+\Gamma^\lambda{}_{\mu\nu}\frac{ds^\mu}{dt}\frac{ds^\nu}{dt}+2\Gamma^\lambda{}_{\mu\bar\nu}\frac{ds^\mu}{dt}\frac{d\bar s^\nu}{dt}=\rho(t)\frac{ds^\lambda}{dt}$$

with a complex factor $\rho(t)$, and in a Kaehler metric the third term on the left drops out. If t is the geodesic length, then for real-valued $\rho(t)$, $\rho(s)$ becomes 0, as is well known, and the curve is a geodesic. In general it need not be so, but $\rho(s)$ is pure imaginary always. If for an Hermitian and a Kaehler metric on the same space the holomorphically flat curves are the same, then the first metric is likewise Kaehler, and if on a compact Kaehler space the scalar curvature is non-negative or non-positive throughout but not $\equiv 0$, then it cannot be in such a correspondence with another Kaehler space whose Ricci curvature is null. Finally, if on Fubini space with curvature k (>0 or <0) one introduces homogeneous coordinates $s^\lambda=\zeta^\lambda/\zeta^0$ for which

$$\zeta^0\overline{\zeta^0}+\tfrac{1}{2}k(\zeta^1\overline{\zeta^1}+\cdots+\zeta^n\overline{\zeta^n})=1,$$

then a curve is holomorphically flat if and only if it can be represented in the form $\xi(t)=\eta+\sigma(t)\zeta$ for some fixed points η, ζ and a complex-valued $\sigma(t)$. *S. Bochner.*

have

★Schouten, J. A. Ricci-calculus. An introduction to tensor analysis and its geometrical applications. 2d ed. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd X. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. xx+516 pp. Broschiert, DM 55.00; ganzleinen, DM 58.60.

Since the publication of the author's Der Ricci-Kalkül [Springer, Berlin, 1924] the subject of tensor calculus as applied to differential geometry has grown very consider-

ably, and the present edition is more than a mere translation of the first. The author has played an active part in the development of the subject and results originally published in papers have also appeared in books written by the author in collaboration with others. The greater part of the work now appears however in its proper setting, and for the first time we have a comprehensive account of the work to date in certain well-defined fields. The only topic in the second edition which has been given less space than in the first is Pfaff's problem, and this is because a full account has already appeared in the book "Pfaff's problem and its generalizations" [Oxford, 1949; MR 11, 179] by the author and van der Kulk.

One valuable contribution of the present volume is the separation of ideas that were previously intermingled but are now seen to belong to essentially different branches of the subject. Another concerns technique and notation; these have been developed and improved considerably since the appearance of the first edition, and the author, who is particularly interested in such matters, now presents the subject in what he considers to be the clearest and most economical form.

There are eight chapters, each comprising from eight to twelve sections, and their contents are briefly as follows.

Ch. I: Algebraic preliminaries. This is confined to the study of n-dimensional number-space subject to a general affine transformation. Vectors and tensors are defined by means of their laws of transformation, and the usual algebra of tensors, multi-vectors, etc., is constructed. The chapter ends with a useful discussion of abridged notation and the various symbolisms of E. Cartan, Clebsch-Aronhold, Weitzenböck, etc.

Ch. II: Analytic preliminaries. An n-dimensional geometric manifold is introduced as an arithmetic manifold together with allowable co-ordinate systems and the pseudo-group of co-ordinate transformations. Geometric objects are defined in the classical way by means of their laws of transformation, and the tangent space of a point appears as the vector space determined by co-ordinate differentials. Sections are given to associated manifolds, systems of differential equations, differential operators, Pfaff's problem, and to Lie and Lagrange derivations with all the formal identities associated with them. Finally, Cartan's suffix-free calculus is described and the correspondences between this and the classical tensor calculus are set out.

Ch. III: Linear connexions. These are introduced in order to provide linear differential operators which give tensor fields when they operate on tensor fields. They in turn lead to the concepts of parallel displacement and parallel tensor fields and to the definition of torsion. The Riemann-Christoffel curvature tensor for a general connexion appears from a consideration of the effect of parallel transport round an infinitesimal closed curve, and the usual identities are established. Geodesics for a general connexion are defined as the autoparallel curves, and normal co-ordinate systems are defined in relation to these curves. A useful section on Fermi co-ordinates is included, and the chapter finishes with an account of some general connexions, including the general Cartan connexion.

Ch. IV: Lie groups and linear connexions. A finite-dimensional Lie group is defined as an analytic group on a space assumed to be covered by a single co-ordinate system, so that this chapter is mainly concerned with what would now be described as the local group; properties of Lie groups in the large are not considered explicitly. Many of the results, especially those concerned with the $(\pm)$-connexions naturally associated with a Lie group, are due to Cartan and the author, and their development in this chapter follows closely along the lines of the original papers, with, at times, a somewhat improved notation. Discussion is almost entirely confined to generalities, but the chapter concludes with a brief account of integrable (solvable) groups and of simple and semi-simple groups.

Ch. V: Imbedding and curvature. This is given to the study of one space imbedded in another, and of the structure induced into the sub-space. Various structures are considered and the often complicated equations relating the curvature tensors of a space and its sub-space are derived. These include the generalisation (to Riemannian sub-spaces) of the classical Frenet formulae for a twisted curve.

Ch. VI: Projective and conformal transformations of connexions. This is concerned with special relations between symmetric connexions, and particularly with projective and conformal transformations and connexions. In addition to the usual properties of projective and conformal spaces there is included an account of some recent work on sub-projective connexions and Adati's problem, and on concircular transformations.

Ch. VII: Variations and deformations. This gives a detailed account of infinitesimal deformations of spaces and sub-spaces and of the (local) theory of groups of motions of a given space. The subject is brought up to date with a discussion of the holonomy groups associated with various connexions. Most of the recent work on special problems in this field is excluded, but full and valuable lists of references are given. The chapter ends with an account of motions and holonomy treated by Cartan's method.

Ch. VIII: Miscellaneous examples. This includes brief but fully annotated accounts of such special spaces as harmonic spaces, spaces with unitary connexions (complex spaces), and spaces of recurrent curvature.

The book concludes with an 87-page bibliography (with authors in alphabetical order) and an index. There is also, at the beginning, a full and helpful list of contents.

Because of the large number of topics included in this book some accounts are necessarily brief, with details of calculations omitted, and very little is included in the way of illustrative examples; for this reason it should perhaps be regarded as a reference book rather than a text book. To compensate for this, detailed references to the literature are given on every topic included in the book, and mention is made of all relevant work which has been excluded. In his preface the author gives a list of those parts of differential geometry which are completely excluded, the most important of these being almost certainly the union of the subject with topology, where local theories are extended to manifolds in the large. Also excluded is all the work on generalised spaces, such as Finsler and the generalised path spaces, and also such subjects as contact transformations, linear elements and connexions of higher order, K-spreads, and applications of geometry to differential equations and applied mathematics.

The book has been extremely well produced; type has been well chosen and the printing is perfectly clear, a feature which is particularly important in work which involves such complicated tensor expressions. *A. G. Walker.*

NUMERICAL AND GRAPHICAL METHODS

*Tablicy integralov Frenelya. [Tables of Fresnel integrals.] Izdat. Akad. Nauk SSSR, Moscow, 1953. 269 pp. (2 inserts). 23.50 rubles.

The functions

$$S(x)=\int_0^x \sin \tfrac{1}{2}\pi t^2 dt, \quad C(x)=\int_0^x \cos \tfrac{1}{2}\pi t^2 dt$$

are tabulated to 7 decimals for $x=0(.001)25$ in the main table. Second differences are given for both functions. Near the origin both functions are small, especially $S(x)$, and so two short tables are given which provide floating decimal values of 7 significant digits. These are of $S(x)$ for $x=0(.001).581$ and $C(x)$ for $x=0(.001).101$. The main table is more than ten times as extensive as previously published tables of the Fresnel integrals. *D. H. Lehmer.*

*Tablicy integral'nogo sinusa i kosinusa. [Tables of the sine and cosine integral.] Izdat. Akad. Nauk SSSR, Moscow, 1954. 473 pp. (2 inserts). 43.75 rubles.

The functions

$$\mathrm{Si}(x)=\int_0^x t^{-1}\sin t\,dt, \quad \mathrm{Ci}(x)=\int_\infty^x t^{-1}\cos t\,dt$$

are tabulated to 7 decimals for

$$x=0(.0001)2(.001)10(.005)100.$$

Second differences are given when they are not negligible. The table for $x\leq 10$ is a contraction of the Nat. Bur. Standards 10-decimal tables of these functions [New York, 1940; MR 2, 239, 366] for the same arguments. For $10<x<100$ the new argument interval has been halved by interpolating in the NBS table [New York, 1942; MR 4; 89]. A nomogram and a table for interpolating with second differences is supplied on two separately inserted cards. For those who need no great accuracy the present volume is a useful condensation of the earlier three volumes. The printing and paper are of good quality. *D. H. Lehmer.*

Polylogarithms. Part I: Numerical values. By the staff of the computation department. Math. Centrum Amsterdam. Rekenafdeling. Rep. R 24, 52 pp. (1954).

The function tabulated in this preliminary report is the analytic function

$$F_\nu(z)=\sum_{n=1}^\infty z^n/n^\nu \quad (|z|<1),$$

called the polylogarithm of order ν. The function $-F_\nu(1-z)$ is known as Spence's transcendent. There are three tables giving values of $F_\nu(z)$ for z along the real axis, along the imaginary axis, and around the unit circle. More exactly, $F_\nu(z)$ is tabulated to 10 decimals for $\nu=1(1)12$ and for $z=x=-1(.01)1$, $z=iy$, $y=0(.01)1$, and $z=\exp\{2\pi ik/400\}$ $(k=0(1)200)$. *D. H. Lehmer* (Berkeley, Calif.).

*Dörrie, Heinrich. **Praktische Algebra.** Verlag von R. Oldenbourg, München, 1955. viii+259 pp. DM 24.00.

This is an elementary practical book which will please readers of a level higher than those for whom it was initially intended. After four introductory chapters concerning the fundamental theorem of algebra, division, symmmetric functions and special equations follow four chapters on estimating and computing roots of algebraic equations. In particular there is a chapter concerning polynomials with only real zeros. The book concludes with a set of 102 problems with solutions, many concerning algebraic identities and special polynomials with real zeros. Some of these results, although not original, may still be unfamiliar to many readers. *O. Taussky-Todd* (New York, N. Y.).

Horner

*Ostrowski, A. **On two problems in abstract algebra connected with Horner's rule.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 40–48. Academic Press Inc., New York, 1954. $9.00.

The application of Horner's method (for the calculation of the value of a polynomial) by automatic computing machines is investigated. There are, in particular, discussions of the problem as to whether less than n multiplications will suffice for the evaluation of an arbitrary polynomial of degree n and also whether a process which employs only n multiplications reduces to Horner's rule. Additions and multiplications by numerical constants are not counted. Both problems are formulated in abstract terms. The first question is answered negatively for $n\leq 4$ and the second question affirmatively for $n\leq 3$. *O. Taussky-Todd.*

Stelson, H. E. Finding the root of an equation by iteration. Skand. Aktuarietidskr. 37, 10–18 (1954).

In this paper is explained a method for the numerical determination of a real root of an equation $x=f(x)$ by means of iteration. From an initial approximation x_0 to the root, one can compute $x_1, x_2, x_3, \cdots$, from $x_{n+1}=f(x_n)$. Then, by means of Newton's interpolation formula with divided differences,

$$\begin{aligned}f(x)=f(x_0)&+(x-x_0)f(x_0,x_1)\\&+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+\cdots\\&+(x-x_0)(x-x_1)\cdots(x-x_{n-1})f(x_0,x_1,\cdots,x_n)\\&+(x-x_0)(x-x_1)\cdots(x-x_n)f^{n+1}(\xi)/(n+1)!\end{aligned}$$

where ξ is some value in the interval including all the arguments $x_0, x_1, \cdots, x_n$, and x, the value of the root is computed. Graphically the method shows how the root is approached and how at each stage limits are obtained between which the true root lies. *E. Frank* (Chicago, Ill.).

Zeuli, Tino. Perfezionamento del metodo di iterazione per la ricerca delle radici reali delle equazioni o dei sistemi di equazioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 88, 259–264 (1954).

Given the equation $x=\phi(x)$, and a first estimate x_1 to a root X, then $x_2=[\phi(x_1)-m_1x_1]/(1-m_1)$, where $m_1=\phi'(x_1)$, gives a better approximation to the root, and $x_2\leq X\leq x_1$. Repetition of the process leads to a sequence $x_2, x_3, x_4, \cdots$, rapidly convergent to X. The formula is extended to a method of iteration of a system of three equations in three unknowns. *E. Frank* (Chicago, Ill.).

Todd, John. The condition of certain matrices. II. Arch. Math. 5, 249–257 (1954).

[For part I see Quart. J. Mech. Appl. Math. 2, 469–472 (1949); MR 11, 619.] The author is concerned with the problem of determining the "condition" of the operator arising from the differential equation $y^{iv}=ky$ as well as a generalization to partial differential equations of the fourth order. In a previous paper he considered the corresponding case for second derivatives. He replaces the differential equation by a different system and shows that the conditional number of the resulting system is $O(n^4)$. The same

result obtains for the fourth-order partial differential equation that is considered. His results are also valid when the fourth derivative is replaced by any even-ordered derivative. A comparative study of some of these problems was made on the National Bureau of Standards SEAC machine.

H. H. Goldstine (Princeton, N. J.).

Ludwig, Rudolf. Über Iterationsverfahren für Gleichungen und Gleichungssysteme. II. Z. Angew. Math. Mech. **34**, 404–416 (1954). (English, French and Russian summaries)

This is a continuation of a previous paper with the same title [same Z. **34**, 210–225 (1954); MR **16**, 78] extending to systems the results previously obtained for single equations. Given the system $f(x)\equiv Hx-g=0$, where f, x and $g(x)$ are vectors, $H(x)$ a matrix, one can ask for matrices $P(x)$ and $Q(x)$ such that the iteration suggested by $(PH+Q)x=Pg+Qx$ has convergence of a given order. Newton's method falls out quite naturally and other second-order iterations are exhibited. Iterations of the third order are much more complicated, as one might expect. Consideration is given to conditions for convergence when the initial approximation lies in a given neighborhood of the true solution. Also considered are methods of combining iterations to obtain an iteration of higher order.

A. S. Householder (Oak Ridge, Tenn.).

Sesini, Ottorino. Interpretazione meccanica ed applicazioni estensive del procedimento "escalator." Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 747–759 (1953).

The "escalator" formulae of J. Morris [The escalator method in engineering vibration problems, Wiley, New York, 1947; MR **9**, 382] for obtaining the characteristic roots of a symmetric $(n+1)\times(n+1)$ matrix A_{n+1} in terms of those of the matrix A_n obtained by deleting the $(n+1)$st row and column from A_{n+1}, are derived by considering conservative linear vibrating systems of mass particles under suitable constraints.

C. R. De Prima.

Pozzolo Ferraris, Giulia. Costruzione grafica della tangente a notevoli curve piane. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **88**, 318–325 (1954).

Ehrich, Fredric F. Differentiation of experimental data using least squares fitting. J. Aero. Sci. **22**, 133–134 (1955).

Frei, Tamás. Anwendung der Momente der Integralkurven zur numerischen Lösung von Differentialgleichungen. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 395–414 (1954). (Hungarian. Russian and German summaries)

The author investigates numerical solutions of initial-value problems for ordinary differential equations $y'=f(x,y)$ by n-term recursion relations of the form

$$y_k=\sum_{i=1}^{n}\alpha_i f_{k-i}+\sum_{i=1}^{n}\beta_i y_{k-i};$$

the first n values are supposed to be computed independently. The motivation for these formulas are mechanical-quadrature formulas applied to the integral relations obtained by multiplying the differential equation by some weight factor (usually a polynomial in x) and integrating between $k-n$ and k. The author observes that

$$|\alpha_i f_y(x_{k-i},y_{k-i})+\beta_i|$$

is an upper bound for the rate of growth of the round-off error, and therefore recommends the use of formulas which minimize this. Some examples are given which compare favorably with answers given by standard methods (Adams-Nyström, Runge-Kutta). It is not made quite clear how to apply the method systematically. The method can also be applied to solve the initial value problem for systems, and to two-point boundary-value problems and eigenvalue problems. The results of examples presented by the author are better than what the Rayleigh-Ritz method gives with about the same amount of labor.

P. D. Lax.

Faedo, Sandro. I metodi ispirati a quello di Ritz nel calcolo delle variazioni e nella teoria delle equazioni differenziali. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17(86)**, 291–302 (1953).

An expository paper, without proofs, concerning direct variational methods of Ritz, Picone [Rend. Circ. Mat. Palermo **52**, 225–253 (1928)] and the author [Rend. Mat. e Appl. (5) **6**, 73–94 (1947); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 466–470 (1953); Ann. Scuola Norm. Sup. Pisa (3) **7**, 91–132 (1953); MR **9**, 106; **15**, 67, 540] for the solution of boundary-value problems for ordinary and partial differential equations.

C. R. De Prima.

Lax, Peter D. Weak solutions of nonlinear hyperbolic equations and their numerical computation. Comm. Pure Appl. Math. **7**, 159–193 (1954).

The author considers first-order hyperbolic systems of the form

$$U_t+F_x+B=0, \tag{1}$$

where U, F, B are vectors, F, B being functions of x, t, U (generally nonlinear). He discusses "weak" solutions, which in effect satisfy (1) except along certain curves, across which they satisfy "jump conditions". The author suggests that the weak solutions can be obtained by solving (1) using finite-difference methods and letting the mesh-dimensions go to zero in a certain way; see also von Neumann and Richtmyer [J. Appl. Phys. **21**, 232–237 (1950); MR **12**, 289], and earlier work by von Neumann referred to there. The author presents much numerical evidence to support his conjectures, and sketches various analytical results concerning weak solutions (to be published elsewhere).

M. A. Hyman (Pittsburgh, Pa.).

Kamynin, L. I. On a defect of the method of lines. Dokl. Akad. Nauk SSSR (N.S.) **95**, 13–16 (1954). (Russian)

The "method of lines" for handling initial-value problems for linear partial differential equations (with constant coefficients) in x and t, with $-\infty<x<+\infty$, consists in replacing the given problem by the solution of the infinite system of ordinary differential equations which arises when the x derivatives in the given partial differential equation are replaced by finite differences, while the t derivatives are retained. The defect of the method of lines consists in that the conditions on the growth (as $|x|\to\infty$) of the initial data and of the solution, which are needed for the uniqueness of the solution of the system of ordinary differential equations are characteristic of the system, and not of the partial differential equation giving rise to it. For example, the initial-value problem for the wave equation $\partial^2u/\partial t^2=\partial^2u/\partial x^2$, with initial conditions $u(x,0)=\varphi_1(x)$, $\partial u(x,0)/\partial t=\varphi_2(x)$; $-\infty<x<+\infty$, leads to the following system of ordinary

differential equations:

$$\frac{\partial^2 u^{(h)}(x,t)}{\partial t^2}=\frac{1}{h^2}\{u^{(h)}(x+h,t)-2u^{(h)}(x,t)+u^{(h)}(x-h,t)\},$$

$$\frac{\partial^l u^{(h)}(x,t)}{\partial t^l}=\varphi_l(x);\ l=0,1;\ x=\cdots,-2h,-h,0,h,2h,\cdots.$$

Now, while the given initial-value problem has a unique solution for all sufficiently smooth initial values, the system obtained by applying the method of lines has a unique solution only in the class of functions which satisfy the restrictive growth conditions:

$$\left|\frac{\partial^l u^{(h)}(x,t)}{\partial t^l}\right|=O\left(\left[(1-\epsilon)\frac{2|x|}{h}\right]!\right);\quad l=0,1;$$

$$|t|\leqq T<+\infty;\quad 0<\epsilon\leqq 1.$$

J. B. Diaz (College Park, Md.).

***van den Dungen, F. H. Sur l'intégration numérique des équations aux dérivées partielles.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 61–70. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The author suggests replacing the partial differential problem

$$(1)\qquad \begin{gathered}\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\quad(-\infty<x<+\infty,\ t\geqq 0)\\ u(x,0)=F(x),\quad \frac{\partial u}{\partial t}(x,0)=G(x)\end{gathered}$$

by the differential-difference problem

$$(2)\qquad \begin{gathered}\frac{d^2 v_h}{dt^2}=\left(\frac{c}{\Delta x}\right)^2[v_{h+1}-2v_h+v_{h-1}]\quad(|h|=0,1,2,\cdots;\ t\geqq 0)\\ v_h(0)=F(x_h),\quad \frac{dv_h(0)}{dt}=G(x_h),\end{gathered}$$

where $v_h(t)\equiv v(x_h,t)$, $x_h\equiv h\Delta x$. He remarks that this will allow using electrical networks (such as differential analyzers) to get the solution approximating u. [For related ideas, see Crank and Nicolson, Proc. Cambridge Philos. Soc. 43, 50–67 (1947), and references given there; MR 8, 409.] For simple choices of F, G, the author solves (2) explicitly, using Fourier transforms. As by-products he obtains certain well-known identities for sums of even-ordered Bessel functions and a sum-approximation to a definite integral. *M. A. Hyman* (Pittsburgh, Pa.).

Richter, A. Zweidimensionale nichtstationäre Felder der Wärmeleitungsgleichung. Graphische Integrationsmethoden zur Lösung parabolischer Differentialgleichungen mit drei unabhängigen Veränderlichen. Arch. Electrotechnik 41, 258–281 (1954).

The author considers graphical solution of time-variant heat-conduction problems in two space dimensions, with the appropriate sorts of initial and boundary conditions. Heat sources are allowed, as are non-isotropy and non-homogeneity of the conducting material. Thus the most general equation considered is

$$\frac{\partial\theta}{\partial t}=P\frac{\partial^2\theta}{\partial\xi^2}+Q\frac{\partial\theta}{\partial\xi}+R\frac{\partial^2\theta}{\partial\eta^2}+S\frac{\partial\theta}{\partial\eta}+T,$$

where P, Q, R, S are functions of ξ, η (which need not be cartesian coordinates) and T is a function of ξ, η, t. The author replaces derivatives in the partial differential equation and boundary conditions by differences in the usual ways; he then exhibits nomograms which allow the resulting "stepping-ahead" formulas to be evaluated, thus giving the solution θ at any point ξ, η with increasing time t. In general, $\Delta\xi$ and $\Delta\eta$ vary and the mesh-ratios $\Delta t/\Delta\xi^2$, $\Delta t/\Delta\eta^2$ have their maximum values consistent with the convergence and stability of the difference solution. The author gives several examples, including solution of a problem involving eddy-currents. *M. A. Hyman* (Pittsburgh, Pa.).

Crandall, Stephen H. Numerical treatment of a fourth order parabolic partial differential equation. J. Assoc. Comput. Mach. 1, 111–118 (1954).

The author studies the numerical solution of the beam-vibration equation

$$(*)\qquad \frac{\partial^4\psi}{\partial x^4}+\frac{\partial^2\psi}{\partial t^2}=0.$$

$\partial^2\psi/\partial t^2$ is replaced by a central second-difference in the usual way, involving times $t-\Delta t$, t, $t+\Delta t$. One may replace $\partial^4\psi/\partial x^4$ by the central fourth-difference at time t ("explicit" difference-equation) or its arithmetic average at times $t-\Delta t$, $t+\Delta t$ ("implicit" difference-equation). For a simple case, the author compares the solution D of (*) with the "explicit" and "implicit" difference-solutions E and I. The implicit scheme, which is not much more difficult to use in this case, gives somewhat better results. The author makes interesting use of vector-diagrams to examine the effect on convergence and stability of the various "modes" which can occur in D, E, and I; he points out that improved convergence can sometimes be obtained by changing the "time-scale" of the difference-solution. *M. A. Hyman.*

Woods, L. C. A note on the numerical solution of fourth order differential equations. Aero. Quart. 5, 176–184 (1954).

The author replaces (1) $\Delta\Delta w=K(x,y)$, where Δ is the Laplacian operator in (x,y)-space, by the system (2) $\Delta w=\rho$, $\Delta\rho=K$. He then solves (2) by relaxation methods, taking account of higher-order corrections to the usual simple difference approximations for equations (2) and the boundary-conditions. This procedure appears to involve much less work than attacking (1) directly by relaxation methods, especially when there are many mesh-points. A simple case is worked out, and the results agree very well with an analytical and a relaxation solution of (1). The author also discusses briefly the numerical determination of steady viscous flow in two dimensions, where the governing differential equations are of the form (2), but K is a function of grad ψ and grad ρ. *M. A. Hyman.*

***Curry, H. B. The logic of program composition.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 97–102. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

The author is concerned in this paper with certain extensions of the concepts introduced by the reviewer and von Neumann [Planning and coding of problems for an electronic computing instrument, Inst. Advanced Study, Princeton, 1948; MR 10, 329] in a series of papers. Principally he introduces the notion of an automatic code for arithmetic programs (the paper although just published is stated to

have been written in 1950) and in general he treats the notion of program composition. *H. H. Goldstine.*

Goodell, John D. The relations between logical, mathematical and computing machine systems. J. Comput. Systems 1, 243–254 (1954).

Samelson, Klaus, und Bauer, Friedrich L. Massnahmen zur Erzielung kurzer und übersichtlicher Programme für Rechenautomaten. Z. Angew. Math. Mech. **34**, 262–272 (1954). (English, French and Russian summaries)

The authors are concerned generally with the structure of programs for automatic computers. They are interested in analysing such structures in detail with the aim of reducing their length and simplifying their form so as to render programming as short and as easy as possible. To this end they discuss quite completely the nature of subroutines and the principles of variable changes. They distinguish three types of such changes and consider how to automatize these changes. The paper closes with an example to illustrate their methods. *H. H. Goldstine.*

Clippinger, R. F., Dimsdale, B., and Levin, J. H. Automatic digital computers in industrial research. V. J. Soc. Indust. Appl. Math. 2 (1954), 184–200 (1955).

For part IV see same J. 2, 113–131 (1954); MR **16**, 180.

Have

★Lehmann, N. Joachim. Bericht über den Entwurf eines kleinen Rechenautomaten an der Technischen Hochschule Dresden. Bericht über die Mathematiker-Tagung in Berlin, Januar, 1953, pp. 262–270. Deutscher Verlag der Wissenschaften, Berlin, 1953. DM 27.80.

★Raymond, F. H. Le calcul analogique. Principes et contribution à une théorie générale. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo, no. 391. Casa Editrice Libreria Rosenberg & Sellier, Torino, 1954. 134 pp. 2000 Lire.

This is an account of lectures given by the author at the Istituto Nazionale per le Applicazioni del Calcolo in Rome, 1952. The paper deals mainly with electronic analogue computers. Ch. 1 is devoted to the principles of analogue computation; elementary mathematical operations and examples of simple analogue computers are described. Ch. 2 refers to algebraic equations and to linear differential equations, especially to those with constant coefficients. A general treatment of electronic differential analysers follows in ch. 3. The last ch. 4 discusses accuracy and stability of analogue computers. The presentation is clear and interesting; all the various aspects of electronic analogue computers are discussed. The booklet is richly equipped with photographs and diagrams of American and European computers. *H. Bückner* (Schenectady, N. Y.).

Have

★Soroka, Walter W. Analog methods in computation and simulation. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. xii+390 pp. $7.50.

Analogue methods in computation have a vastly extended literature. Some 30 years ago patent specifications were the main source of information. Since that time and especially since the development of differential analysers the interest in analogue computers has steadily grown. Nowadays they represent a large scientific field with a variety of interesting features due to the fact that so many single-purpose computers exist and that widely different ways for their design are possible. This situation makes it an appreciable task to give a survey of the analogue methods. This book covers the wide field of both single- and general-purpose computers and of the various possibilities for their design, especially mechanical and electrotechnical ones.

Chapter 1 deals with mechanical computing elements. The author deals with elements for addition and subtraction, scale factors, questions of accuracy, mechanical integrators. The use of integrators for multiplication and for the generation of special functions is described together with other methods of multiplication. The chapter also includes squaring, division, differentiation and the application of servomechanisms for the inversion of functions. Mainly the same problems are treated in chapter 2, this time with reference to electrical and electronic computing elements. Examples are: High-gain negative-feed-back amplifiers as useful components of electronic integrators and derivators; potentiometer curve fitting and selsyn-type transformers for the generation of sinusoidal functions. Machines for solving simultaneous algebraic equations are dealt with in chapter 3. Mechanical equation solvers as well as electrical ones are described; they refer mainly to linear equations. A discussion of secular equation computers links this chapter to the next one, where non-linear equations are considered in general; special points are how to compute the zeros of polynomials, harmonic synthesizers, application of electromagnetic fields and of impedance networks, the latter with respect to partial fractions. Chapters 5 and 6 are devoted to differential analysers in general, and the discussion covers both mechanical and electronic instruments, particularities of their components and typical applications. Dynamical analogies are the subject of chapter 7. The analogy between mechanical and electrical elements is pointed out. Vibrating systems and lumped systems of many degrees of freedom follow. Applications refer particularly to the bending of beams, to plasticity and to questions of creep. Chapter 8 covers circuits for finite differences, taken from ordinary or partial differential equations. A wide field of applications is mentioned, e.g. Laplace's equation and problems of elasticity. The last chapter 9 deals with membrane and conductive sheet analogies.

The book is well balanced, as both theoretical discussions and questions of practical interest are dealt with. It will be useful for the designer of analogue computers and for the scientist who just wants to use them. It can also be recommended to students as most of the deductions are made "ab ovo". *H. Bückner* (Schenectady, N. Y.).

Bashe, C. J., Buchholz, W., and Rochester, N. The IBM Type 702, an electronic data processing machine for business. J. Assoc. Comput. Mach. 1, 149–169 (1954).

Cohen, Arnold A. Remington Rand arithmetic calculators and installations for automatic calculation and control. Calc. Automat. Cibernet. 3, no. 8, 33–39 (1954). (Spanish)

★Abdel-Messih, Moheb Aziz. Zur Theorie der Rechengeräte mit linearen Potentiometern. Dissertation, Eidgenössische Technische Hochschule in Zürich, 1954. 77 pp.

This paper presents formulae and graphs of the functions which can be represented as the output potential of circuits involving one or more potentiometers and fixed resistances. The settings of these potentiometers, some of which may be ganged, are regarded as independent variables. The possibilities for one and two variables and a limited number of

potentiometers are covered. The objective is the representation of functions in computing devices. The load is considered part of the circuit, and the representation indicated by Thévenin's Theorem is not given. *F. J. Murray.*

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*Torres-Quevedo, Gonzalo. **Les travaux de l'école espagnole sur l'automatisme.** Les machines à calculer et la pensée humaine, pp. 361–381. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

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*Torres-Quevedo, Gonzalo. **Présentation des appareils de Leonardo Torres-Quevedo.** Les machines à calculer et la pensée humaine, pp. 383–406. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

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*McCulloch, W. S., Lettvin, J. Y., Pitts, W. H., et Dell, P. C. **Une comparison entre les machines à calculer et le cerveau.** Les machines à calculer et la pensée humaine, pp. 425–443. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

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*Uttley, A. M. **Réalisation de suites d'opérations logiques au moyen des machines à calculer.** Les machines à calculer et la pensée humaine, pp. 465–473. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

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*Andreev, P. P. **Proof of a theorem by means of the principle of correspondence.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 46–50. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

*Kreĭnes, M. A. **On a property of surfaces nomographable in single-valued functions.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 96–97. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

*Golovina, S. M. **A nondeterminantal method of construction of nomograms.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 98–106. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Textbooks on nomography show that the three functions of the scales of a nomogram must satisfy the Massau determinant. This determinant may be used in the design of the nomogram. However, simpler techniques are frequently possible. In the present work, the relation between the points of a circle and their stereographic projection onto a tangent straight line is used to display several commonly sought nomographic relations. [See Whittaker and Robinson, Calculus of observations, 3rd ed., Blackie, London-Glasgow, 1940, p. 128.] Projective transformations of these nomograms give other conic sections which, for some purposes, have more desirable scales. *M. Goldberg.*

*Nalband'yan, A. **A problem on the construction of a graph of circles.** Nomografičeskiĭ sbornik [Nomographic collection], pp. 123–124. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

A quantity which is a quadratic function of two independent variables is representable as a set of curves on a Cartesian grid. By the appropriate transformation of the Cartesian variables, the curves may be transformed into circles. This transformation is derived. Such transformations are given also by M. d'Ocagne [Calcul graphique et nomographie, 2ème éd., Doin, Paris, 1914, pp. 193–201].

M. Goldberg (Washington, D. C.).

ASTRONOMY

Vernić, Radovan. **Bahnen des restringierten Dreikörperproblems dargestellt im Inertialsystem.** Jugoslav. Akad. Znan. Umjet. Rasprave. Odj. Mat. Fiz. Tehn. Nauke 1, 81–123 (1952). (Serbo-Croatian. German summary)

Let two particles P_1, P_2 of masses $m_1=m_2=\frac{1}{2}$ (the total mass being the unit of mass) revolve in a circular orbit about their common centre of mass with constant angular velocity $n=1$. Consider a third infinitesimal particle P which moves in the plane of motion of P_1 and P_2 in such a way that, while it is subject to the Newtonian attractions of P_1 and P_2, it does not disturb the Keplerian motion of the two bodies. The resulting model is called the Copenhagen (E. Strömgren) form of the restricted problem of three bodies.

Consider two barycentric coordinate systems: (x, y), called an inertial coordinate system, and (ξ, η), a rotating coordinate system which rotates uniformly with angular velocity $n=1$ in the positive sense so that the ξ-axis always coincides with the line determined by the position of the particles P_1 and P_2. During the period 1895–1935 several hundreds of orbits of P were calculated numerically at the Copenhagen Observatory. Among them are 63 simply periodic and 13 periodic-asymptotic orbits with respect to the rotating (ξ, η)-system.

The main results of the present paper are: (i) with reference to an inertial coordinate system the orbits in general are not closed curves but have a general resemblance to an ellipse with a progressive motion of the apsides caused by the incommensurability of the angular velocity of both the finite particles and the mean motion of the infinitesimal particle. As a consequence of this the dynamical system under consideration, instead of having periodic motions will in general have only Poisson stability in the sense of Poincaré; (ii) the only motions, periodic with respect to both barycentric coordinate systems, are the linear oscillations of the infinitesimal particle.

Since in the transfer from the rotating ξ, η-coordinate system to an inertial coordinate system the main properties of the orbits (such as periodicity, cusps and loops) are not preserved, the author concludes that the knowledge of the totality of orbits in the Copenhagen restricted three-body problem cannot constitute the key to the understanding of the general three-body problem. This is not meant to underestimate the methods used and the results obtained by Strömgren and his collaborators. *E. Leimanis.*

Nadile, Antonio. Sopra un caso particolare del problema ristretto dei tre corpi. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 98–118 (1953).

The case here considered of the restricted problem of three bodies is that in which the distance of the infinitesimal body from the center of gravity of the two finite bodies is large compared with the distance between the two finite bodies. In setting up the equations of motion, third and higher powers of the ratio of this latter distance to the former are neglected. As a first approximation, the infinitesimal body describes a Keplerian ellipse with focus at the center of gravity of the other two bodies. The so-called second approximation, obtainable essentially as variations of the first approximation, is available in terms of certain quadratures in accordance with well known methods. The second approximation is considered in some detail in the two following cases: (1) The Keplerian ellipse of the first approximation is a circle. (2) The ratio of the angular velocity of the two finite bodies to the mean motion of the infinitesimal body (in the first approximation) is an integer. *D. C. Lewis.*

Černyi, Sergeĭ D. Free nutation of the earth. Acad. Serbe Sci. Publ. Inst. Math. 6, 57–62 (1954). (Russian)

The author assumes the earth to be a rigid body with three unequal principal axes of inertia. He solves the equations of rotational motion under inertia alone and obtains an expression for the angle between the axis of rotation and the axis of the greatest moment of inertia. Upon adaptation of a number of different sets of observational data to his formulae he finds the value 332.2 days as the period of free nutation for the earth. *R. G. Langebartel.*

Woolley, R. v. d. R. A study of the equilibrium of globular clusters. Monthly Not. Roy. Astr. Soc. 114, 191–209 (1954).

For a spherically symmetric system of stars describing Keplerian orbits about a central mass, it is shown that stars of different masses are governed by independent distribution functions each of the form

$$(*)\qquad \nu_m(v, r)=v^2F_m(v^2-2\phi),$$

where v denotes the absolute velocity of the star. In (*) ν_m denotes the number of stars of mass m and velocity v (per unit interval of m and v) at a distance r from the center where the gravitational potential is ϕ. Various special forms of F_m are considered; and the consequences of combining the resulting expressions for the density (obtained by integrating ν_m over v and r) with Poisson's equation for ϕ are deduced. *S. Chandrasekhar* (Williams Bay, Wis.).

Burbidge, G. R. On the dynamical stability of magnetic stars. Astrophys. J. 120, 589–595 (1954).

In this paper the magnetic energy associated with the decay field of a star is evaluated. Two cases are considered: the case of a homogeneous conductor and a special case of variable conductivity considered by Wrubel [Astrophys. J. 116, 291–298 (1952); MR 15, 751]. The maximum polar field compatible with the requirement of S. Chandrasekhar and E. Fermi [ibid. 118, 116–141 (1953); MR 15, 168] that the magnetic energy of an equilibrium configuration be less than the numerical value of the gravitational potential energy is estimated for a number of special cases.

S. Chandrasekhar (Williams Bay, Wis.).

Prendergast, Kevin H. One dimensional self-gravitating star systems. Astr. J. 59, 260–262 (1954).

The general solution of the steady state one-dimensional Liouville and Poisson equation system is obtained under assumption of sufficiently rapid vanishing of the phase space density distribution function at infinity in both variables. The method is applied to determine the explicit expression for the distribution function of the form of the Dirac δ-function of a linear function of the energy.

R. G. Langebartel (Urbana, Ill.).

Pogorzelski, W. Problème du mouvement stationnaire dans une couche gazeuse rayonnante. Bull. Acad. Polon. Sci. Cl. III. 2, 7–8 (1954).

A summary of an investigation to be published in detail later. A plane-stratified atmosphere is in motion perpendicularly to the planes of stratification and radiation is also passing through it in all directions. A formal solution of the equations governing the steady state of such a system, using non-linear integral equations, is described and it is stated that the results could be evaluated by successive approximations. *G. C. McVittie* (Urbana, Ill.).

Thüring, Bruno. Methodologische Untersuchungen zur Kosmologie. Methodos 6, 95–113 (1954).

RELATIVITY

Rubin, Herman, and Suppes, Patrick. Transformations of systems of relativistic particle mechanics. Pacific J. Math. 4, 563–601 (1954).

This paper contains an axiomatic foundation of relativistic particle mechanics. It starts with a set of kinematical and dynamical axioms, which defines a system of relativistic particle mechanics (S.R.P.M.). Then the notion of generalized Lorentz-transformation is introduced and it is shown that such a transformation carries an S.R.P.M. into an S.R.P.M. These generalized Lorentz-transformations appear to be the only transformations (in a wide class) with this property. The investigation is similar to that of classical particle mechanics by McKinsey, Sugar and Suppes [J. Rational Mech. Anal. 2, 253–272, 273–289 (1953); MR 14, 1023]. *J. Haantjes* (Leiden).

Bogorodskiĭ, A. F. The equivalence principle and field equations of the general theory of relativity. Kiïv. Derž. Univ. Publ. Kiïv. Astr. Obs. 1948, no. 2, 23–29 (1948). (Russian) first

Using the ~~fine~~-order approximation for the line element in the presence of a system of particles, the author verifies that the total energy is conserved, where the latter is given by the sum of the rest-energy, the kinetic energy, and the

gravitational potential energy of the particles. However, he finds that when he takes as the equation of motion of a test particle the equation of the geodesic, he is unable to put it into the form of the Newtonian equation of motion with a gravitational potential determined by the total energy of the system. He therefore concludes that the equivalence of mass and energy does not hold in the general relativity theory insofar as gravitational mass is concerned. It should be pointed out that the paper appears to contain calculational errors. *N. Rosen* (Haifa).

Bogorodskiĭ, A. F. The integration of the field equations for a system of point masses. Kiïv. Derž. Univ. Publ. Kiïv. Astr. Obs. **1948**, no. 2, 31–45 (1948). (Russian)

The Einstein gravitational field equations are solved for the case of a system of particles by expanding the field components in a series of small quantities. There appear to be errors in the calculation. *N. Rosen* (Haifa).

Hlavatý, Václav. Report on the recent Einstein unified field theory. Rend. Sem. Mat. Univ. Padova **23**, 316–332 (1954).

This paper gives a clear and useful summary, without proofs, of the main results of the thirteen earlier papers in which the author (in one case in collaboration with A. W. Sáenz) has examined Einstein's unified field-theory [see MR **13**, 687, 774, 994; **14**, 213, 416, 505, 1132; **15**, 654; **16**, 408]. *H. S. Ruse* (Leeds).

De Simoni, Franco. Le soluzioni generali della statica a simmetria sferica nell'ultima teoria unitaria di Einstein. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 348–355 (1954).

The author finds the general static spherically symmetric solutions of Einstein's 1953 unified field equations [The meaning of relativity, 4th ed., Princeton, 1953, Appendix II; MR **14**, 805]. The solutions, which are found by the method of Papapetrou [Proc. Roy. Irish Acad. Sect. A. **52**, 69–86 (1948); MR **10**, 580], Wyman [Canad. J. Math. **2**, 427–439 (1950); MR **13**, 79] and Bonnor [Proc. Roy. Soc. London. Ser. A. **209**, 353–368 (1951); **210**, 427–434 (1952); MR **13**, 695] include as special cases the solutions found by these authors for earlier versions of the theory. Physical significance is not discussed in the present paper. *F. A. E. Pirani* (Dublin).

Tonnelat, Marie-Antoinette. Validité de la solution générale des équations, d'Einstein $g^{\underline{\mu\nu}}_{+-}$; $\rho=0$ dans le cas $\varphi=0$. C. R. Acad. Sci. Paris **239**, 1468–1470 (1954).

Let $g_{\lambda\mu}$ be the basic tensor of the Einstein unified theory and denote by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ the determinants of $g_{\lambda\mu}$, $g_{(\lambda\mu)}$ and $g_{[\lambda\mu]}$. The solution $\Gamma'_{\lambda\mu}$ of the basic equations as given by the author [C. R. Acad. Sci. Paris **230**, 182–184 (1950); J. Phys. Radium (8) **12**, 81–88 (1951); **13**, 177–185 (1952); MR **11**, 569; **13**, 79; **14**, 213] involves the inverse tensor to $g_{[\lambda\mu]}$. Hence it would seem that this solution holds only for $\mathfrak{k}\neq 0$. Fortunately, the inverse tensor appears in the corresponding formula only in the combination $\partial\sqrt{\mathfrak{k}}/\partial g_{[\lambda\mu]}$ which obviously exist even if $\mathfrak{k}=0$. [For the solution in the exceptional cases $\mathfrak{g}/\mathfrak{h}=0$, 2 see V. Hlavatý, Elementary basic principles of the unified theory of relativity, B_3; to appear in J. Rational Mech. Anal.] *V. Hlavatý* (Bloomington, Ind.).

Ikeda, Mineo. On static solutions of Einstein's generalized theory of gravitation. I. Progr. Theoret. Phys. **12**, 17–30 (1954).

Since Einstein first proposed his most recent unified field theory several people have pointed out that there are many possibilities for the choice of a skew-symmetric tensor F_{ij} which will represent the electromagnetic field. In the present paper the author investigates a choice of F_{ij} which satisfies one of Maxwell's equations and is derivable from a vector potential ϕ_i. Further, for a certain class of coordinate systems it is shown that the boundary conditions imposed on ϕ_i are invariant. Unfortunately, however, this class of coordinate systems is only a sub-group of the general group of four-dimensional transformations.

Under the physical interpretation of the field quantities described above the static spherically symmetric solutions of the field equations are studied and it is shown that satisfactory physical interpretations are obtained. *M. Wyman* (Edmonton, Alberta).

Buchdahl, H. A. Reciprocal static solutions of the equations $G_{\mu\nu}=0$. Quart. J. Math., Oxford Ser. (2) **5**, 116–119 (1954).

The author considers an n-dimensional metric space whose line element is given by $ds^2=g_{\mu\nu}dx^\mu dx^\nu$. Further a solution of the equations $G_{\mu\nu}=0$, where $G_{\mu\nu}$ is the Ricci tensor, is called static if for a fixed integer "a",

$$g_{ia}=0 \quad (i=1, 2, \cdots, a-1, a+1, \cdots, n)$$

and $\partial g_{\mu\nu}/\partial x^a=0$. Under these conditions the author proves that if $g_{\mu\nu}$ is any static solution of these equations then $\mathfrak{g}_{\mu\nu}$ defined by $\mathfrak{g}_{ij}=(g_{aa})^{2/(n-3)}g_{ij}$ $(i, j=1, 2, \cdots, a-1, a+1, \cdots, n)$, $\mathfrak{g}_{ia}=0$ $(i=1, 2, \cdots, a-1, a+1, \cdots, n)$, and $\mathfrak{g}_{aa}=(g_{aa})^{-1}$ is also a solution. One example on the use of the method is given. *M. Wyman* (Edmonton, Alberta).

Mavridès, Stamatia. Le choix de la métrique et du champ électromagnétique en théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **239**, 637–640 (1954).

The author considers some of the implications of two different choices of the field quantities which are to be used to determine the metric tensor and electromagnetic field tensor in Einstein's unified field theory. *M. Wyman.*

Infeld, L. On the motion of bodies in general relativity theory. Acta Phys. Polon. **13**, 187–204 (1954). (Russian summary)

The author takes up again the problem of the derivation of the equations of motion of a particle from the field equations of general relativity. A simpler and briefer method than those hitherto used is devised, which may be regarded as a synthesis of the methods formerly employed by Einstein, Infeld, Hoffmann, Fock and Papapetrou. More than one particle is envisaged, the system of particles being described by an energy-tensor expressed in terms of the masses and velocities of the particles, together with the Dirac δ-function. Thus the energy-tensor is zero everywhere in the space-time, except in the small regions occupied by the particles. The latter are assumed to move slowly and therefore the time-derivatives of the gravitational potentials can be regarded as being of a higher order in small quantities than their spatial derivatives. The potentials and the energy-tensor are expanded as power series in terms of a parameter λ, which is described as being small. Small compared with unity is presumably meant though no standard of comparison is expressly mentioned. A critical step in the expansions appears to be the assumption that the mass of any one of the particles is a power series in λ^2 of the form $m=\lambda^2m_2+\lambda^4m_4+\cdots$. The conclusions that then follow from the field equations are: (a) working to the order λ^3, the equations of motion of a particle reduce to the statement that

for each particle m_1 is constant; (b) working to order λ^4, the Newtonian equations of motion are obtained; (c) non-Newtonian equations of motion are found by proceeding up to the order λ^6. It is also stated that, by choosing a suitable coordinate system, the equations of motion given are the exact equations.

In general relativity, other kinds of approximation procedures in terms of G/c^2 or of $1/c^2$ (G is the gravitational constant, c, the velocity of light) are employed. It would be interesting to know how these methods are related to Infeld's but this is at present not clear because no physical interpretation of the parameter λ is given. *G. C. McVittie.*

Gupta, Suraj N. Gravitation and electromagnetism. Phys. Rev. (2) 96, 1683–1685 (1954).

The author recalls some features of his method of quantization of the gravitational field of general relativity [Proc. Phys. Soc. Sect. A. 65, 608–619 (1952); MR 14, 417], in which Einstein's theory is regarded as a flat-space theory with a Lagrangian density containing an infinite number of terms. This method yields gravitational quanta (gravitons) of zero rest-mass and spin 2.

He now inverts the argument, considering a field of neutral particles of zero rest-mass and spin 2. If this field can interact with others, then its source must be a symmetric tensor of zero divergence. The only such tensor readily available is the energy-momentum tensor, but for consistency this must be the energy-momentum tensor of all fields, including the original spin-2 field. The non-linearity of the equations for the spin-2 field follows at once from this, and a natural method of constructing the Lagrangian of this field requires that the latter have an infinite number of terms.

The author attributes this characteristic of the Lagrangian to the fact that the field it describes is of spin 2. It appears to the reviewer that it is the non-linearity, arising when a field contributes to its own source function, and not the spin, which is responsible for the infinite number of terms in the Lagrangian.

F. A. E. Pirani (Dublin).

Costa de Beauregard, Olivier. Diffraction par une ouverture d'Univers tridimensionnelle plane du genre temps. C. R. Acad. Sci. Paris 240, 160–162 (1955).

Fabre, Hervé. L'action photonique en gravitation et en cosmologie. C. R. Acad. Sci. Paris 240, 158–160 (1955).

MECHANICS

***Artobolevskiĭ, I. I., Zinov'ev, Vyač. A., i Èdel'štein, B. V. Sbornik zadač po teorii mehanizmov i mašin.** [**Collection of problems on the theory of mechanisms and machines.**] **2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1951. 195 pp. 4.40 rubles.

Many recent Russian textbooks on mechanisms do not have problems needed for the engineering student. The present work fulfills this need. Answers are given in the back of the book and, depending upon the demands of the problem, are qualitative, numerical or graphical. The arrangement is by topic, following the works of the senior author. Each set of problems is preceded by a brief discussion of the general principles. The following topics suggest the scope of the work: kinematic pairs, composition of mechanisms, classification, degrees of freedom, trajectories, velocity and acceleration diagrams, centrodes, toothed mechanisms, design of mechanisms and cams, forces, friction, balancing of rotating mechanisms. *M. Goldberg.*

Levitskiĭ, N. I., and Šahbazyan, K. H. The synthesis of spatial four-link mechanisms with lower pairs. Trudy Sem. Teor. Mašin i Meh. 14, no. 54, 5–24 (1954). (Russian)

The four-bar mechanism made of two skew shafts carrying cranks joined by a connecting-rod through ball-and-socket joints can be specified by eight parameters. By analytical geometry, the relations between these parameters and the variable angles are derived. These equations may be used for approximating a desired curve by calculating the required parameters to pass the curve through selected points or by other curve-fitting processes (least squares or other criteria). The equations are used in a numerical example in which eight parameters are calculated. Special cases are considered in which fewer parameters, down to only three, are subject to calculation. *M. Goldberg.*

Bottema, O. On Alt's special three-bar sectic. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indag. Math. 16, 498–504 (1954).

If P_1, P_2 and P_3, three positions of a point in a plane body which moves over a fixed plane, have a circumscribing circle of constant radius, the locus of the center of the circle is a sextic curve which Alt [Z. Angew. Math. Mech. 1, 373–398 (1921)] called an R_M curve. Alt showed that this curve can be generated by a point on the connecting-rod of a coupler mechanism. Recent papers by Schuh [Nederl. Akad. Wetensch. Proc. Ser. A. 57, 92–103, 129–139, 140–151, 238–249, 250–262 (1954); MR 16, 62] considered the locus K of a point whose pedal triangle with respect to given triangle ABC has a circumscribed circle of constant radius. Schuh's curve is identical with Alt's curve. Bennett [Proc. London Math. Soc. (2) 20, 59–84 (1921)] showed how to generate a sextic curve from a general plane cubic. It is shown that the sextic constructed from the cubic by Bennett's procedure is an Alt curve if, and only if, the three asymptotic lines of the cubic are the sides of an equilateral triangle. Also, the asymptotic lines are tangent to the circumcircle of triangle ABC. *M. Goldberg.*

Prosciutto, Aristide. Per una teoria geometrica unitaria degli ingranaggi per assi sghembi. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 9, 39–47 (1952).

The transmission of uniform motion between skew non-intersecting shafts is accomplished by means of hyperboloidal (hypoid) gears mounted on these shafts. The base surface of each gear is a regulus (a ruled hyperboloid of revolution) whose axis is the axis of the shaft. The line of contact between these surfaces is a straight-line generator of the surface. For positive drive, gear teeth of arbitrary shape are added to one surface and these generate the mating teeth on the other surface. The same general method is used for deriving the points of the mating base surface, the tooth

surface and the tooth contact line. Ordinary spur gears and bevel gears are then only special cases of hyperboloidal gears.
M. Goldberg (Washington, D. C.).

Mewes, E. Darstellung räumlicher Kräftesysteme durch rechtwinklige Kraftkreuze. Ing.-Arch. 22, 348–356 (1954).

Jeffreys, Harold. What is Hamilton's principle? Quart. J. Mech. Appl. Math. 7, 335–337 (1954).

This is a criticism of a recent paper by R. S. Capon [same J. 5, 472–480 (1952); MR 14, 917], in which it was asserted that Hamilton's principle is not satisfied by non-holonomic systems. As indicated in the review of Capon's paper, it has always been obvious that this assertion was to be accepted only in a very restricted sense. The present author summarizes his views in the following words: "In non-holonomic systems the so-called work function does not express the whole of the forces acting; the Hölder-Voss modification of Hamilton's principle allows correctly for this fact."
D. C. Lewis (Baltimore, Md.).

Pars, L. A. Variation principles in dynamics. Quart. J. Mech. Appl. Math. 7, 338–351 (1954).

This is also a criticism of the paper by R. S. Capon [same J. 5, 472–480 (1952); MR 14, 917]. The author remarks that Hamilton's principle can be generalized in at least two different ways each of which is equivalent to the original principle when the system is holonomic, but only one of which leads to the correct equations of motion for non-holonomic systems. The proper form for Hamilton's principle (for both holonomic and non-holonomic systems) is established and its relation to other variational principles is discussed. *D. C. Lewis* (Baltimore, Md.).

Malgarini, Giorgio. Sopra l'attrito ed i relativi principi meccanici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 17(86), 223–257 (1953).

This work constitutes a generalization of some ideas of P. Appell, later developed further by Vâlcovici [Acad. Roum. Bull. Sect. Sci. 26, 440–452 (1946); MR 9, 629], whereby the forces of sliding friction are included in a generalized principle of d'Alembert. In the present paper more general types of friction are considered, as are also generalized coordinates and such things as the principle of least action. *D. C. Lewis* (Baltimore, Md.).

Zeuli, Tino. Sopra alcuni casi di riducibilità alle quadrature per le equazioni del moto di un punto sollecitato da forze posizionali. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 21–42 (1953).

A general sufficient condition is developed under which it is possible to construct a linear transformation which reduces a system of the form $\ddot{x}=F(x)$, where x and F are n-vectors, to another system of the same form with the variables separated in such a manner that integration by quadrature is trivially possible. *D. C. Lewis.*

Armstrong, H. L. A way of representing the motion of a classical system as being along a geodesic in space-time. Amer. J. Phys. 22, 615–617 (1954).

The Jacobian form of the principle of least action is concerned with the path of the system rather than with its motion in time. In this paper it is suggested that, by adding a degree of freedom in uniform motion to serve as a clock and extending the configuration space to a space-time by adding the coordinate of that degree of freedom, the motion is completely described as being along a geodesic. The equations of motion obtained in this way are identical with those derived from Hamilton's principle. Two examples are given. *J. Haantjes* (Leiden).

Aržanyh, I. S. Dynamical systems of rank greater than zero. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 5, 96–110 (1949). (Russian)

Out of a welter of misprints, misstatements, and redundant notation, the reviewer has gleaned something that is submitted here as a paraphrase of the paper. Let K_ρ, L_ρ, $\rho=1, \cdots, r$, be functions of t, q_i, $\dot{q}_i$. Put $p_i=K_\rho\partial L_\rho/\partial\dot{q}_i$ (sum over repeated indices), and consider "the dynamical system of rank r" whose equations of motion are $\dot{p}_i=K_\rho\partial L_\rho/\partial q_i$. Obviously, if $\delta t=0$,

$$K_\rho\delta L_\rho=\delta(p_i\dot{q}_i)+\dot{p}_i\delta q_i-\dot{q}\delta p_i.$$

Now choose H_ρ so that $K_\rho(\delta L_\rho+\delta H_\rho)=\delta(p_i\dot{q}_i)$. (The author gives the impression that $H_0=-L_0+p_i\dot{q}_i$, $H_\rho=-L_\rho$ for $\rho>0$ is a unique choice. He also assumes $K_0=1$.) Then $\dot{p}_i\delta q_i-\dot{q}_i\delta p_i=-K_\rho\delta H_\rho$, and one obtains "the canonical equations of rank r" (*) $\dot{q}_i=K_\rho\partial H_\rho/\partial p_i$, $-\dot{p}_i=K_\rho\partial H_\rho/\partial q_i$. Three proofs of this "important" theorem are given, whereupon it is shown that every set of ordinary differential equations of even order can be given the form (*). In view of this, the theorem that (*) are preserved by contact transformations is anticlimactic. There follow several pages of involved computations leading to the law of transformation of K_ρ and H_ρ in terms of Jacobi's form [cf. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed., Cambridge, 1937, p. 309] with the result that only H_0 changes according to the familiar law [loc. cit.]. Since the H_ρ are not uniquely determined, the significance of this statement is not obvious. *A. W. Wundheiler.*

Savin, G. N. On dynamic forces in a shaft lifting cable (lifting a load). Dokl. Akad. Nauk SSSR (N.S.) 97, 991–994 (1954). (Russian) $T_1 = \zeta Q$

A lifting cable hangs over a revolving cylinder with fixed axis whose circular velocity is given. At the lower end of the cable a load of weight Q is attached resting on a fixed base. Let the force applied to the lower end of the cable before lifting the load be $T_0=\zeta Q$, where $0\leqq\zeta\leqq1$. As the cylinder begins to revolve, the lifting cable will first stretch out during an interval of time τ, the attached load remaining fixed on the support for $\zeta\neq1$. The load will start to move upward only after the force applied to the lower end of the lifting cable attains the value Q. The cable is considered as an elastic-viscous string, i.e. its stiffness at bending is neglected.

Instead of the equations of motion for an element of the cable and the attached load, considered in a special case by Neronov [Akad. Nauk SSSR. Prikl. Mat. Meh. (N.S.) 1, 91–116 (1937)], the author introduces an integro-differential equation (containing a parameter n) which is equivalent to the above mentioned equations in the sense that any solution of these equations satisfying the prescribed boundary conditions satisfies also this integro-differential equation. The converse theorem, however, is in general not true. By means of this integro-differential equation the author then constructs approximate solutions of the problem under consideration.
E. Leimanis (Vancouver, B. C.).

Chrapan, J. An explicit solution of the motion of a spherical pendulum using the Jacobian transcendental functions. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4**, 55–69 (1954). (Slovak. Russian summary)

Defrise, Pierre. Analyse géométrique de la cinématique des milieux continus. Inst. Roy. Météorol. Belgique. Publ. Ser. B. no. 6, 63 pp. (1953).

The author's objective appears to be the formulation of Euclidean kinematics in geometric language, using a four-dimensional geometry in which the fourth coordinate is distinguished by special properties. The reviewer was not able to discern any results not already well known in other forms. The presentation may appeal to those who learn modern higher geometry before becoming familiar with the elements of mechanics. *C. Truesdell.*

Hydrodynamics, Aerodynamics, Acoustics

Edwards, S. Sherman. The unsteady forces on a circular cylinder in the presence of two symmetrically disposed line vortices. J. Aero. Sci. **22**, 72 (1955).

Bloh, È. L. On the impact of an ellipsoid of revolution floating on the surface of a quite heavy fluid. Prikl. Mat. Meh. **18**, 631–636 (1954). (Russian)

The author considers an ellipsoid E of revolution, half immersed in an ideal fluid which fills a lower half-space and which is initially at rest. One of the principal axes of E is to be orthogonal to the surface of the fluid. A prescribed initial velocity is then imparted to E by means of an impulsive force. The velocity distribution in the fluid at the moment of impact is studied in the limiting case that the density of the fluid is arbitrarily large. The procedure involves only a formal modification of the method used by the author in two related papers [same journal **17**, 579–592, 705–726 (1953); MR **16**, 82], the boundary condition $\varphi=0$ for the velocity potential at the free surface of the fluid being here replaced by the condition $\partial\varphi/\partial n=0$. The solution again appears as a series of associated Legendre functions of the first and second kinds. The physical justification of the boundary condition at the free surface, both in the present work and in the above cited earlier papers of the author, is not clear to the reviewer. *R. Finn.*

Lance, G. N., and Deland, E. C. On the differential analyser solution of the water bells problem. Proc. Phys. Soc. Sect. B. **68**, 54–55 (1955).

Kostyukov, A. A. Resistance of bodies in a fluid to motion near a vertical wall. Dokl. Akad. Nauk SSSR (N.S.) **99**, 349–352 (1954). (Russian)

The author writes down the integral for the wave resistance of a submerged body moving at constant velocity parallel to a rigid wall in a heavy fluid of finite depth. He notes that the resistance (for a body with the same source distribution) is always less than twice that when the wall is absent. The case of submerged source and dipole is also treated, including the lateral force. The author is apparently unaware of Lunde's paper [Trans. Soc. Naval Arch. Marine Engrs. **59**, 25–76 (1951)] which treats exhaustively many problems of this sort [see pp. 44–50, 57–59].

J. V. Wehausen (Providence, R. I.).

Collins, W. D. A note on Stokes's stream-function for the slow steady motion of viscous fluid before plane and spherical boundaries. Mathematika **1**, 125–130 (1954).

In the case of simple axisymmetrical Stokes flows, it is shown that the stream function for the viscous flow, bounded by either a sphere, a plane or a combination of both, can be expressed in terms of a stream function for an unbounded irrotational motion, such a uniform flow, a source, a doublet and so on. *Y. H. Kuo* (Ithaca, N. Y.).

Dean, W. R. Note on the motion of an infinite cylinder in rotating viscous liquid. Quart. J. Mech. Appl. Math. **7**, 257–262 (1954).

G. I. Taylor has shown that the two-dimensional motion of an infinite cylindrical solid, immersed in incompressible, inviscid fluid, under the action of assigned external forces is not altered by a uniform rotation of the whole system (including the external forces) about an axis parallel to the generators of the cylinder, provided the density of the liquid is the same as that of the solid and the motion of the solid is measured relative to the system. The author here reviews Taylor's analysis and then shows that it can be extended with very little change to a viscous liquid. The reviewer notes that this result explains, at least in part, the striking agreement between Taylor's theory and experiment.

G. W. Morgan (Providence, R. I.).

Mitchner, Morton. An approximate solution to the Navier-Stokes equations. Quart. Appl. Math. **12**, 401–404 (1955).

Soit V' une solution exacte non permanente des équations de Navier-Stokes appartenant à la classe des mouvements qui ont été appelés (l'auteur n'emploie pas cette dénomination) mouvements pseudo-plans de deuxième espèce [Berker, Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Taffin-Lefort, Paris-Lille, 1936]. L'auteur tire de toute solution exacte V' pseudo-plane de deuxième espèce une solution approximative V qui est également pseudo-plane de deuxième espèce, l'approximation n'étant valable que dans un intervalle de temps $(t_0, t_0+\epsilon)$. Pour $t=t_0$, V se réduit à la somme du champ V' et d'un écoulement par droites parallèles avec une distribution linéaire de la vitesse; V représente donc l'évolution dans le temps d'une perturbation V' imposée au temps $t=t_0$ à l'écoulement par droites parallèles. L'auteur applique le procédé à un exemple. Il montre que la solution approximative V qui fait l'objet de cet article est différente d'une solution exacte obtenue par le signataire de ce compte-rendu (cf. loc. cit. pp. 39–55) et qui offre une analogie superficielle avec la solution V. *R. Berker* (Istanbul).

Dean, W. R. On the steady motion of viscous liquid past a flat plate. Mathematika **1**, 143–156 (1954).

By calculating the non-linear terms of the Navier-Stokes equations from the Blasius boundary-layer solution ψ_1, the author found a second approximation ψ_2 by solving a linear inhomogeneous 4th order partial differential equation. This solution is finally reduced to a sum of two definite integrals and the Blasius solution. Numerical results show that the ratio ψ_2/ψ_1 is everywhere greater than 1 and reaches the highest value of about 1.2 in the neighborhood of the leading edge.

Y. H. Kuo (Ithaca, N. Y.).

Strang, J. A. Incompressible flow near a solid boundary. Communications Fac. Sci. Univ. Ankara. Sér. A. 6, 51–76 (1954). (Turkish summary)

This paper concerns the reasons for the failure of boundary-layer theory in the neighborhood of separation and possible solutions. In the reviewer's opinion, the reasons offered are irrelevant. The solutions given are formal and, in certain respects, incomplete. It is believed that up to now Dean's solution of the problem [Proc. Cambridge Philos. Soc. 46, 293–306 (1950); MR 11, 697] is most appropriate. *Y. H. Kuo* (Ithaca, N. Y.).

Squire, L. C. Boundary layer growth in three dimensions. Phil. Mag. (7) 45, 1272–1283 (1954).

An analysis is made of the boundary layer in three-dimensional flow immediately following an impulsive start from rest in an incompressible fluid. The argument used by Goldstein and Rosenhead for the analogous plane problem [Proc. Cambridge Philos. Soc. 32, 392–401 (1936)] is easily extended to three-dimensional boundary-layer flow. The second and third approximations are carried out, at least as far as the derivatives at the body surface, which yield the skin friction. The results are applied to an ellipsoid of axis ratios 30:6:1, which resembles a wing in symmetrical flow. The object is to determine the locus in space and time of the phenomenon of separation; this is preceded by a discussion of the criterion for separation in three-dimensional flow. It is concluded from the present calculations that separation first appears at the rear stagnation point at the midsection. This separation point then moves forward and separation begins on neighboring streamlines at points not coincident with the rear stagnation points; meanwhile streamlines farther outboard remain unseparated. The separated area expands outward as time progresses, and at a certain time all streamlines over the ellipsoid encounter separation. The times at which these various events occur are determined. *W. R. Sears* (Ithaca, N. Y.).

Malkus, W. V. R. The heat transport and spectrum of thermal turbulence. Proc. Roy. Soc. London. Ser. A. 225, 196–212 (1954).

Consider a layer of fluid of depth d heated below in which a constant mean adverse temperature gradient $-\beta_0$ is maintained. The net flux of heat, H, across any horizontal layer is the same and can be written as

$$H=\kappa\beta(z)+(\overline{W\Delta T})_z=\kappa\beta_0+\overline{\overline{W\Delta T}},$$

where κ is the thermometric conductivity; $\beta(z)$ is the local mean temperature gradient; $(\overline{W\Delta T})_z$ is the average value at depth z of the product of the normal component of the velocity W, and the departure in the temperature ΔT from the mean value $\bar{T}(z)$ prevailing at z; and $\overline{\overline{W\Delta T}}$ is the average of $\overline{W\Delta T}$ over all z. Expressing W and ΔT as Fourier sine series in $(\pi z/d)$, we can write

$$(1)\quad \overline{\overline{W\Delta T}}=\sum_{n=1}^{\infty}\left\{\alpha_n \sin^2\frac{n\pi z}{d}+\gamma_n \sin\frac{n\pi z}{d}\sin\frac{(n-1)\pi z}{d}\right\}$$

and

$$(2)\quad \overline{\overline{W\Delta T}}=\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n.$$

The author now supposes that the foregoing two series terminate at $n=n_0$; and further that the coefficients α_n and γ_n are to be determined by the postulated condition that $(\overline{W\Delta T})_M/\overline{\overline{W\Delta T}}$ is a minimum (where $(\overline{W\Delta T})_M$ is the maximum value of the convective term in the interval $(0, d)$) with respect to z, the α's and the γ's. The author observes that the latter condition is met for the choice

$$(3)\quad \gamma_n=0;\quad \alpha_n=4\kappa\beta_0\left(1-\frac{n}{n_0+1}\right)\quad\text{and}\quad H=\kappa\beta_0(n_0+1).$$

The corresponding value of $\beta(z)/\beta_0$ is

$$(4)\quad \frac{\beta_{n_0}(z)}{\beta_0}=\frac{1}{n_0+1}\,\frac{\sin^2[(n_0+1)\pi z/d]}{\sin^2(\pi z/d)}.$$

The author then shows by an analysis similar to that used in solving the classical Bénard problem for the onset of convection in a layer of fluid heated below that the condition for the stability of the (n_0+1)th mode with a constant mean temperature gradient $-\beta_0$ (for suitable boundary conditions) is the same as the condition for the stability of a prevailing temperature gradient $-\beta_{n_0}(z)$ given by (4). The author interprets some experimental work of his [same Proc. 225, 185–195 (1954)] in terms of these ideas.

S. Chandrasekhar (Williams Bay, Wis.).

Krzywoblocki, M. Z. E. Bergman's linear integral operator method in the theory of compressible fluid flow. C. Axially symmetric flow and singularities. Österreich. Ing.-Arch. 8, 237–263 (1954).

In this third part [for parts A and B see same Arch. 6, 330–360 (1952); 7, 336–370 (1953); MR 14, 510; 15, 367] of his work on Bergman's linear integral operator method, the author at first discusses axially symmetric flow where the method of infinite approximations by a sequence of linear equations is applied. A discussion of the properties of Bergman's operator, singularities, fundamental solutions and generalization of Blasius' formulae close this part.

Author's summary.

Kaplan, Carl. On the small-disturbance iteration method for the flow of a compressible fluid with application to a parabolic cylinder. NACA Tech. Note no. 3318, 36 pp. (1955).

Cette note technique est consacrée à l'étude des approximations successives pour un écoulement stationnaire bidimensionnel d'un fluide compressible; la fonction de courant et le potentiel des vitesses sont développés en fonction d'un paramètre lié à l'épaisseur jusqu'aux termes d'ordre trois inclus. Une application est faite au cas d'un obstacle ayant la forme d'un cylindre parabolique. Ordonnant le résultat par rapport au carré du nombre de Mach à l'infini amont, l'auteur montre que le résultat obtenu est en accord avec celui obtenu par la méthode de Janzen-Rayleigh. Diverses comparaisons sont faites avec les résultats obtenus avec d'autres méthodes. *P. Germain* (Paris).

Manwell, A. R. A new singularity of transonic plane flows. Quart. Appl. Math. 12, 343–349 (1955).

L'auteur, dans cet article, revient sur le problème de l'impossibilité de l'apparition d'une ligne limite dans les écoulements plans transsoniques [Friedrichs Communications on Appl. Math. 1, 287–301 (1948); MR 10, 638; Morawetz et Kolodner, ibid. 6, 97–102 (1953); MR 14, 1033]. Il cherche à expliquer la possibilité de la non existence d'un écoulement transsonique pour des vitesses importantes par l'apparition éventuelle d'un nouveau type de singularité. Cette singularité est liée à une discontinuité des dérivées le long d'une caractéristique issue d'un point d'une ligne de courant dans le plan de l'hodographe en lequel la tangente

présente une discontinuité. Une telle solution a pour partie principale une solution particulière de Darboux de l'équation de Tricomi, dont l'auteur étudie bien le prolongement dans le demi plan elliptique; par contre il ne porte aucune attention, semble-t-il, à la reflexion éventuelle d'une telle singularité le long de la ligne sonique. Il y aurait lieu, pour le moins, de vérifier qu une telle réflexion ne change pas essentiellement les conclusions de cet article.
P. Germain (Paris).

Tricomi, F. G. Beispiel einer Strömung mit Durchgang durch die Schallgeschwindigkeit. Monatsh. Math. **58**, 160–171 (1954).

L'auteur expose comment l'équation "de Tricomi" joue un role important dans l'étude des écoulements plans transsoniques d'un fluide compressible; il rappelle en particulier l'application qui en a été faite à l'étude approchée de l'écoulement dans une tuyère. *P. Germain* (Paris).

Imai, Isao. On a refinement of the transonic approximation theory. J. Phys. Soc. Japan **9**, 1009–1020 (1954).

L'essentiel de cet article est consacré à l'étude des solutions de l'équation de Tricomi, et très spécialement de celles susceptibles de jouer un role important dans l'étude des écoulements transsoniques par la méthode de l'hodographe. Certes, les variables choisies et les méthodes utilisées pour former les solutions singulières dont on a besoin, ne sont pas exactement les mêmes que celles employées dans les mémoires consacrés à ce sujet déjà très étudié, mais les résultats obtenus sont, comme il se doit, très sensiblement équivalents.

L'originalité de cette étude consiste plutot en l'utilisation possible de ces résultats pour résoudre des problèmes d'écoulements transsoniques. En fait, les équations de l'auteur comprennent à la fois l'interprétation classique valable pour les écoulements presqu'uniformément transsoniques et celle que l'on en déduit par la transformation de Loewner [NACA Tech. Note no. **2065** (1950); MR **13**, 464] étudiée également par Fenain [Rech. Aéro. no. **33**, 11–28 (1953); MR **14**, 1141].

D'autre part, et ceci est complètement nouveau, l'auteur propose d'utiliser non pas les lois du fluide fictif associé à l'équation du type mixte considéré, mais les lois du fluide réel. L'approximation ne consiste plus que dans le remplacement des solutions de l'équation rigoureuse de Chaplygin par les solutions de l'équation de Tricomi. Certes la solution dans le plan physique ne satisfera plus aux lois de conservation de la masse et de la quantité de mouvement, mais ceci, on le comprend, peut être le prix d'une approximation en fait supérieure aux approximations classiques qui viennent d'être rappelées. Toutefois le rapporteur se demande comment, dans ces conditions, s'opère le passage du plan de l'hodographe au plan physique; le présent article ne permet pas de répondre à cette question, qui fera sans doute l'objet d'une prochaine publication. *P. Germain.*

Germain, Paul. General theory of conical flows and its application to supersonic aerodynamics. NACA Tech. Memo. no. **1354**, vii+333 pp. (1955).

Translation of O.N.E.R.A. Publ. no. **34** (1949); MR **12**, 452.

Martin, John C. A vector study of linearized supersonic flow. Applications to nonplanar problems. NACA Rep. no. **1143** (1953), ii+34 pp. (1954).

"Supersedes" NACA Tech. Note no. **2641** (1952); MR **14**, 329.

Roy, Maurice. Formules pour ondes de choc stationnaires en courant plan. C. R. Acad. Sci. Paris **238**, 2369–2372 (1954).

The author derives explicit expressions for the values and/or gradients along the shock of various state variables immediately behind a curved two-dimensional stationary shock. The flow in front of the shock is an isentropic and isoenergetic flow of an ideal fluid and the quantities are obtained in terms of the state variables and their derivatives in front of the shock. In particular, a relatively simple expression has been obtained for the vorticity immediately behind the shock. Some deductions are made from the derived formulae, especially for various special cases; but it should be noted that similar formulae and deductions have been or could be obtained from previous papers on this topic, as for example, C. C. Lin and S. I. Rubinov, J. Math. Physics **27**, 105–129 (1948) [MR **10**, 78]. *P. Chiarulli.*

Munakata, Ken-iti. On the interaction of a plane shock wave with a wedge. J. Aero. Sci. **21**, 501–504 (1954).

This paper is a "preliminary account of an attempt to solve approximately" the pseudo-stationary, non-isentropic flow resulting from the interaction of a plane shock wave of arbitrary shock strength and a wedge of arbitrary vertex angle. The pseudostationary equations of motion are linearized with respect to some reference state and solutions are obtained by separation in polar coordinate variables. The reviewer is dubious about the value of this method. The linearization cannot be expected to apply for the large disturbances implied by arbitrary shock strength and arbitrary vertex angle, and those cases where linearization is valid have been very adequately considered by Bargmann, Lighthill, and others. The linearization itself is done incorrectly, and the infinite series solution obtained by superposition contains arbitrary superposition constants which are to be determined from the unknown form of the reflected shock by "methods of trial and error" or by making "use of experimental results when available." *P. Chiarulli.*

Ludloff, H. F., and Friedman, M. B. Difference solution of shock diffraction problem. J. Aero. Sci. **22**, 139–140 (1955).

Nitsche, Johannes. Über die Fortpflanzung kleiner Störungen in flüssigen Medien, betrachtet als Ausbreitungsvorgang von Unstetigkeiten in den Lösungen der Bewegungsdifferentialgleichung. Z. Angew. Math. Mech. **34**, 361–373 (1954). (English, French and Russian summaries)

Il s'agit dans cet article de la propagation d'une discontinuité issue d'un point dans un milieu constitué de deux fluides différents séparés par une surface plane. Le problème est traité dans le cadre de l'approximation acoustique usuelle. Le problème plan pour un seul milieu est abordé en premier lieu, et les formules générales sont rappelées. On étudie ensuite le cas de deux milieux; le centre de la perturbation étant choisie au voisinage de la surface plane de séparation. L'auteur détermine le choc réfléchi et le choc réfracté ainsi que la répartition des intensités de la discontinuité de pression le long de ces diverses surfaces de choc. Le but de l'article est très spécialement d'étudier ce que deviennent ces résultats lorsque la distance de l'origine de la perturbation à la surface de séparation des deux milieux tend vers zéro, et de mettre en évidence les particularités qui apparaissent dans ce passage à la limite. La fin de l'article est consacré à l'examen de ce problème dans le cas de révolution et au cas de la réflexion totale. *P. Germain.*

Oswatitsch, K., und Sjödin, L. Kegelige Überschallströmung in Schallnähe. Österreich. Ing.-Arch. **8**, 284–292 (1954).

Les auteurs étudient le problème de l'écoulement supersonique d'un fluide compressible autour d'un cone de révolution sans incidence, lorsque la vitesse du fluide est très légèrement supérieure à la célérité du son. L'étude est faite à partir des équations simplifiées des écoulements transsoniques, l'angle du cone étant supposé petit. Comme l'on sait, le problème se ramène à l'intégration d'une équation différentielle qu'il faut effectuer numériquement. La méthode suivie met automatiquement en évidence l'influence du paramètre de similitude des écoulements transsoniques et les résultats viennent très heureusement compléter les résultats très complets obtenus lorsque la vitesse du fluide est nettement supersonique. Les auteurs mettent spécialement en évidence les différences entre ces résultats et ceux de la théorie linéaire. Sous cette forme, les résultats peuvent être étendus à des cones de faible ouverture de forme quelconque, par application d'un théorème d'équivalence de K. Oswatitsch et F. Keune. *P. Germain* (Paris).

D'yakov, S. P. On the stability of shock waves. Ž. Eksper. Teoret. Fiz. **27**, 288–295 (1954). (Russian)

Consider two regions of uniform plane flow antiparallel to the y-axis and separated by a normal shock at $y=0$. Suppose the flow is perturbed so the shock becomes $y=\eta_0 e^{i(kx-\omega t)}$ while the uniform flow upstream remains undisturbed. The author seeks time-dependent solutions of the linearized equations for entropy, velocity, and pressure proportional to $e^{i(kx+ly-\omega t)}$ of which there are two types: entropy-vorticity waves and sound waves. The appropriate linear combination is determined by means of the linearized shock conditions. These and the hydrodynamic equations yield two equations which define ω and l implicitly as functions of k, the parameters of the unperturbed flow, and $(\partial V/\partial p)_H$, where V is specific volume and H indicates differentiation along the Hugoniot curve of possible states p, V behind the normal shock. For a general Hugoniot curve and fixed values of all parameters except ω, l, and $(\partial V/\partial p)_H$ it is shown that if $(\partial V/\partial p)_H$ lies in a certain finite interval the perturbed motion is stable. In an adjacent finite interval the perturbations neither grow nor attenuate, and there is spontaneous emission of sound. For values of $(\partial V/\partial p)_H$ in the two remaining semi-infinite intervals the perturbed motion is unstable. *J. H. Giese* (Havre de Grace, Md.).

D'yakov, S. P. Shock waves in binary mixtures. Ž. Eksper. Teoret. Fiz. **27**, 283–287 (1954). (Russian)

For one-dimensional steady flow the author formulates equations for conservation of mass (for the mixture and for one component), of momentum, and of energy, taking into account viscosity, heat conduction, diffusion, thermal diffusion, and pressure diffusion. He expands the changes of enthalpy and specific volume to terms of second order in the change of pressure p and to first order in the changes of entropy and of the concentration c of one component to obtain an ordinary differential equation that can be solved explicitly for $p(x)$. This yields an estimate of the thickness of the shock and also implies that $c(x)$ rises from its value c_0 ahead of the shock to a maximum and then returns to c_0. Also considered is a suspension of dust particles for which it is found that the shock width is increased, and the concentration of dust particles decreases in the shock.

J. H. Giese (Havre de Grace, Md.).

★Krasil'ščikova, E. A. Krylo konečnogo razmaha v sžimaemom potoke. [The wing of finite span in a compressible flow.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 158 pp. 4.50 rubles.

This is a connected account of seven of the author's previously reviewed papers on linearized supersonic thin-wing theory published between 1947 and 1951 [see MR **9**, 392; **10**, 77; **12**, 216, 797; **13**, 507; **14**, 815; **15**, 910; **16**, 86]. Contents: Chap. I: Unsteady motion of a wing in supersonic flow (17 pp. of formulation of problem and generalities); Chap. II: Steady motion etc. (62 pp.); Chap. III: Harmonic oscillation etc. (42 pp.). The text has been taken practically verbatim from the original papers. Only the introduction and sections of examples and general remarks at the ends of Chaps. II and III seem to be new. To consider only steady motion, for brevity, the main theme of the book is that the velocity potential function satisfies

$$-\pi\varphi(x,y,z)=\iiint \varphi_z(\xi,\eta,0)\times[(x-\xi)^2-k^2(y-\eta)^2-k^2z^2]^{-\frac{1}{2}}d\xi d\eta,$$

where $\varphi_z(x,y,z)=\partial\varphi/\partial z$ and $k^2=M^2-1$. On the wing φ_z is essentially the local angle of attack. In the disturbed region outside the wing $\varphi_z(x,y,0)$ can be determined by solving Abel integral equations. This idea is applied in exhaustive detail to wings of high or low aspect ratio with arbitrary cutouts. The point of view is strictly mathematical; virtually no effort has been made to derive formulas for aerodynamic coefficients. The style and organization of the book are extremely repetitive, a feature which might render it suitable for use as a reference for elementary courses on compressible flow. *J. H. Giese* (Havre de Grace, Md.).

Woolston, Donald S., and Runyan, Harry L. Some considerations on the air forces on a wing oscillating between two walls for subsonic compressible flow. J. Aero. Sci. **22**, 41–50 (1955).

The integral equation previously formulated by Runyan and Watkins [NACA Rep. no. **1150** (1954); MR **16**, 194] is solved approximately by collocation. The effects of variation of Mach number, frequency, and tunnel-height/airfoil-chord ratio on the lift due to a pitching oscillation are calculated. The effect of tunnel walls on the bending-torsion flutter speed of a typical configuration also is presented.

J. W. Miles (Los Angeles, Calif.).

Haskell, R. N., and Grogan, G. C. Slender bodies of low wave drag. J. Aero. Sci. **22**, 138–139 (1955).

Kumar, S., and Tietjens, O. G. A note on the circulation function and the induced efficiency of an eight-bladed propeller. J. Indian Inst. Sci. Sect. B. **37**, 103–107 (1955).

Kaplan, S. A. On the conservation of circulation in magneto-gas-dynamics. Astr. Ž. **21**, 360–361 (1954). (Russian)

It is known in hydrodynamics that the constancy of the circulation (equal to the line interval of dv/dt over a closed contour) implies that grad $\rho\times$grad $p=0$. In hydromagnetics this condition is clearly replaced by

$$\text{grad }\rho\times\text{grad }p-\frac{1}{4\pi}\text{curl}\left\{\frac{\mathbf{H}}{\rho}\times\text{curl }\mathbf{H}\right\}=0.$$

S. Chandrasekhar (Williams Bay, Wis.).

Pai, Shih I. Laminar flow of an electrically conducting incompressible fluid in a circular pipe. J. Appl. Phys. 25, 1205–1207 (1954).

The author shows that the standard equations of hydromagnetics appropriate to viscous flow in a cylinder [cf. Chandrasekhar, Proc. Roy. Soc. London Ser. A. **216**, 293–309 (1953); MR **14**, 813] allow a simple stationary solution when the impressed magnetic field **H** has the form $H_r = -\frac{1}{2}H_0 r$ and $H_z = H_0 z + h(r)$, where H_r and H_z denote the radial and the z-components of **H**; further, H_0 is a constant and $h(r)$ is a function expressible in terms of Bessel functions. [The question of how magnetic fields with radial cylindrical symmetry can be generated is left unexplained.]

S. Chandrasekhar (Williams Bay, Wis.).

Michael, D. H. The stability of an incompressible electrically conducting fluid rotating about an axis when current flows parallel to the axis. Mathematika 1, 45–50 (1954).

The standard equations of hydromagnetics [cf. Chandrasekhar, Proc. Roy. Soc. London. Ser. A. **216**, 293–309 (1953); MR **14**, 813] in the case of zero viscosity and infinite electrical conductivity and appropriate to flow between two rotating cylinders (of radii a and b) allow stationary solutions in which only the transverse components V and H of the velocity and the magnetic field are non-vanishing; also V and H can be arbitrary functions of the distance r from the axis. For first-order perturbations of this solution in which the components of the velocity and the magnetic field in the radial, transverse and z-directions are given, respectively, by

$$[u(r)\cos\lambda z\, e^{i\omega t},\ V(r)+v(r)\cos\lambda z\, e^{i\omega t},\ w(r)\sin\lambda z\, e^{i\omega t}]$$

and $[0, H(r)+h(r)\cos\lambda z\, e^{i\omega t}, 0]$, where λ denotes the wave number of a symmetric disturbance in the z-direction and $e^{i\omega t}$ represents its rate of growth, the equations of motion lead to

$$(*)\qquad \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(ru)\right\} = \lambda^2\left\{1-\frac{g(r)}{\omega^2}\right\}u,$$

where

$$g(r) = \frac{1}{r^3}\frac{d}{dr}(Vr)^2 - \frac{\mu}{4\pi\rho}r\frac{d}{dr}\left(\frac{H_0}{r}\right)^2.$$

Solutions of (*) must be sought which vanish at $r=a$ and b. The problem is thus reduced to a characteristic-value problem of the standard Sturm-Liouville type and the condition for stability is that $g(r)$ is nowhere negative.

S. Chandrasekhar (Williams Bay, Wis.).

Friedlander, F. G. Diffraction of pulses by a circular cylinder. Comm. Pure Appl. Math. 7, 705–732 (1954).

The aim of this paper is to investigate the behaviour of the pressure behind the diffracted wave when a sound pulse is scattered by a rigid circular cylinder, the motion being a two-dimensional one. The motion in such a problem can be defined by means of a velocity potential ϕ; in the usual notation

$$p = \frac{\partial\phi}{\partial t},\quad Ku = -\frac{\partial\phi}{\partial x},\quad Kv = -\frac{\partial\phi}{\partial y},$$

where K is the acoustic impedance; the potential ϕ satisfies

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = q,$$

where q is a source term and the normal derivative of ϕ vanishes on the surface of the cylinder $r=1$.

The first step is to find the Green's function, which satisfies

$$\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = 2\pi\delta(x-x_0)\delta(y-y_0)\delta(t-t_0)$$

and the boundary condition $\partial G/\partial r = 0$ when $r=1$. In polar coordinates

$$G = \sum_{-\infty}^{\infty} F(r, \theta+2m\pi, t; r_0),$$

where F is defined for $-\infty<\theta<\infty$ by the equation

$$\frac{\partial^2 F}{\partial t^2} - \nabla^2 F = \frac{2\pi}{r_0}\delta(r-r_0)\delta(\theta)\delta(t),$$

the singularity being $r=r_0$, $\theta=0$, and the boundary condition $\partial F/\partial r = 0$ when $r=1$.

This problem is solved by introducing the Fourier transform with respect to θ of the Laplace transform of F with respect to t; in symbols

$$\bar F(r,\theta,s;r_0) = \int_0^\infty F(r,\theta,t)e^{-st}dt$$

$$F^*(r,\nu,s;r_0) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\bar F(r,\theta,s;r_0)e^{-i\nu\theta}d\theta.$$

F^* is readily expressed in terms of modified Bessel functions of order ν, viz.

$$F^* = (2\pi)^{-1/2}\frac{K_\nu(sr_0)}{K_\nu'(s)}\{I_\nu(sr)K_\nu'(s) - I_\nu'(s)K_\nu(sr)\}$$

when $1\leq r\leq r_0$; r and r_0 are interchanged when $1\leq r_0\leq r$. It follows that the Laplace transform of G is

$$\bar G(r,\theta,s;r_0) = K_0(sR) - \frac{I_0'(s)K_0(sr_0)K_0(sr)}{K_0'(s)} - 2\sum_1^\infty \frac{I_n'(s)K_n(sr_0)K_n(sr)}{K_n'(s)}\cos n\theta,$$

where R is the distance from $(r_0, 0)$ to (r,θ).

The function $\bar F$ can be written as

$$2\pi\sum_{j=1}^{\infty}\frac{I'_{i\mu_j}(s)K_{i\mu_j}(sr_0)K_{i\mu_j}(sr)}{[\partial K'_{i\mu}(s)/\partial\mu]_{\mu=\mu_j}}e^{-\mu_j|\theta|},$$

where $\nu = i\mu_j$ $(j=1,2,\cdots)$ are the zeros of $K_\nu'(s)$.

Approximate formulae for F, valid when $T=t-\tau$ is large where $\tau = |\theta| - \cos^{-1}r^{-1} - \cos^{-1}r_0^{-1} + (r^2-1)^{1/2} + (r_0^2-1)^{1/2}$, are obtained in terms of the Airy function; but the detailed results are too complicated to quote in a review. The results are applied to the problem of diffraction of plane pulses.

E. T. Copson (St. Andrews).

Elasticity, Plasticity

Babič, V. M. On the solution of Cauchy's problem for a system of equations of the theory of elasticity of a non-homogeneous elastic medium. Dokl. Akad. Nauk SSSR (N.S.) 96, 1125–1128 (1954). (Russian)

The author shows that Hadamard's method of constructing fundamental solutions of systems of partial differential equations agrees with that of S. L. Sobolev [some applications of functional analysis in mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565]. Study of two-

and three-dimensional problems of the theory of elasticity of nonhomogeneous media illustrates the argument.
J. R. M. Radok (Melbourne).

Mossakovskiĭ, V. I. A fundamental mixed problem of the theory of elasticity for a half-space with a circular curve of separation of the boundary conditions. Prikl. Mat. Meh. **18**, 187–196 (1954). (Russian)

The mixed boundary-value problem considered is that of determining the displacements and stresses in the elastic half-space $z \leqq 0$ in (x, y, z)-space when the displacements u, v, w are prescribed throughout the circle $x^2+y^2 \leqq a^2$ in the plane $z=0$, and the external stress components σ_z, τ_{zx}, τ_{yz} are prescribed throughout $x^2+y^2 \geqq a^2$ in the plane $z=0$. By seeking the displacements in the form [see E. Trefftz, Handbuch d. Physik, Bd 6, Springer, Berlin, 1928]:

$$u=\varphi_1+z\frac{\partial\psi}{\partial x},\quad v=\varphi_2+z\frac{\partial\psi}{\partial y},\quad w=\varphi_3+z\frac{\partial\psi}{\partial z},$$

$$\frac{\partial\psi}{\partial z}=\frac{1}{4\nu-3}\left(\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y}+\frac{\partial\varphi_3}{\partial z}\right),$$

where φ_1, φ_2, φ_3, ψ are harmonic functions, the problem is reduced to the determination of harmonic functions under suitable mixed boundary conditions. As an example, the problem of the impact of a plane circular rigid punch on an elastic half-space is worked out. *J. B. Diaz.*

Nowacki, Witold. Thermal stresses in anisotropic bodies. I. Arch. Mech. Stos. **6**, 481–492 (1954). (Polish. Russian and English summaries)

The author deduces the Betti-Rayleigh reciprocal theorem for a nonhomogeneous anisotropic elastic body subjected to nonuniform heating and obtains the formulas for the mean deformations in the heated body. *I. S. Sokolnikoff.*

Lur'e, A. I. The stressed state in an elastic cylinder loaded on the lateral surface. Inžen. Sb. **17**, 43–58 (1953). (Russian)

The problem described in the above title has a long history and was investigated by many workers. One of the first, Lamé, solved the problem of an infinite cylinder loaded along a line on the lateral surface on its entire length. The author justifies his contribution claiming not only a new method but also some new results. He considers an infinite cylinder loaded normally on the lateral surface in the following ways: (a) continuous constant loads along a line segment of finite length, (b) continuous constant loads along a circumference of a normal section, (c) loads arbitrarily applied. The author deals mainly with case (a). Solutions (displacements in this case) are in form of Fourier integrals which are transformed by contour integration into functions in series form. When the length of the loaded segment increases, solutions approach Lamé solutions. This work provides a convenient means of finding an error when Lamé's method is used as an approximation. At the end the author shows how to apply solution of case (a) to solve case (b), and then how to apply (a) and (b) to solve (c). The author mentions in the preface that in principle his method could be extended to a hollow cylinder but it would not be practical. *T. Leser* (Lexington, Ky.).

Tolokonnikov, L. A. Finite symmetric deformations of a strip. Prikl. Mat. Meh. **18**, 619–626 (1954). (Russian)

The author discusses bending of a cylindrical block of elastic material into a cylindrical block. The general form of the equations which he attempts to solve is not given, but his analysis suggests that he assumes incompressibility and that the strain energy is a quadratic function of the principal extensions. If this be the case, he has not satisfied all relevant equations, though one can obtain a solution bearing some resemblance to his from a general solution given by Adkins, Green and Shield [Philos. Trans. Roy. Soc. London. Ser. A. **246**, 181–213 (1953); MR **15**, 369].
J. L. Ericksen (Washington, D. C.).

Acharya, Y. V. G., and Srinath, L. S. Determination of principal stresses in an isotropic material under conditions of plane strain. Appl. Sci. Res. A. **5**, 45–54 (1954).

The authors consider a two-dimensional state of stress where the maximum shearing stress is assumed to be known. The resulting statically determined problem of finding the stress components is hyperbolic and the characteristics coincide with the shear lines. In a photoelastic test the maximum shearing stress and the shear lines could be determined experimentally. Hence, as the authors illustrate by an example, the characteristic equations of the above problem could be used to calculate the stress components in such a test. The paper also contains similar remarks which concern the plane flow of a perfectly plastic material whose yield stress is a function of the position. *E. T. Onat.*

Gulkanyan, N. O. On torsion of a prism of triangular cross-section. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Nauk **6**, no. 5–6, 69–76 (1953). (Russian. Armenian summary)

An approximate solution of the Saint Venant torsion problem for a beam with a triangular cross section is deduced when the section is an isosceles or a right triangle. The boundary conditions are satisfied approximately by minimizing the square of the error in the assumed series solution. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Nowiński, Jerzy. The torsion of a rectangular rod in which one cross section remains plane. Arch. Mech. Stos. **5**, 47–66 (1953). (Polish. English summary)

An approximate solution is obtained for the torsion problem of a rectangular rod, one of whose cross-sections is constrained to remain plane. The method used is that proposed by S. P. Timoshenko [Theory of elasticity, McGraw-Hill, New York, 1934, p. 302] for the solution of the corresponding problem for a narrow rectangular beam.
I. S. Sokolinikoff (Los Angeles, Calif.).

Nowacki, Witold. On certain cases of torsion of bars. Arch. Mech. Stos. **5**, 21–46 (1953). (Polish. English summary)

An exact solution is obtained for the Saint Venant torsion problem of orthotropic bars for the following cross-sections: 1) a rectangle with narrow slits, 2) a section whose components are rectangles, 3) an annular sector, 4) a circle with rectilinear or curvilinear slits. The problem is formulated with the aid of the membrane analogy and the solution is deduced from the appropriate Fredholm integral equations of the first kind. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Čankvetadze, G. G. Bending of a circular plate supported at several points. Inžen. Sb. **14**, 73–80 (1953). (Russian)

The author finds a solution of the problem of determining the bending of a thin circular plate which is uniformly loaded perpendicularly to the plane of the plate, and which

is supported at a finite number of points of its periphery. It is stated that this paper was submitted for publication in 1951, before the appearance of the paper on a similar subject by M. M. Fridman [Prikl. Mat. Meh. 15, 258–260 (1951); MR 13, 510]. Some problems of this type had been considered earlier by A. Nádai [Phys. Z. 23, 366–376 (1922); Die elastische Platten . . . , Springer, Berlin, 1925] by a seemingly more laborious method. The results are compared with those of A. I. Lurye [Prikl. Mat. Meh. 4, 93–102 (1940)] and S. P. Timoshenko [Theory of plates and shells, McGraw-Hill, New York, 1940]. *J. B. Dias.*

Kalandiya, A. I. On the problem of equilibrium of an elastic plate with supported boundaries. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 193–210 (1953). (Russian. Georgian summary)

The author extends work by Halilov [Prikl. Mat. Meh. 14, 405–414 (1950); MR 12, 185; 14, 1037] to multiply connected regions. Under the same general assumptions, referring to the boundaries and the types of solutions sought, the existence of solutions is proved by reduction of the problem to a system of singular integral equations.

J. R. M. Radok (Melbourne).

Burmistrov, E. F. On the concentration of stresses about oval openings of a certain form. Inžen. Sb. 17, 199–202 (1953). (Russian)

The author investigates the stresses around oval and almost rectangular holes in isotropic plates subject to uniaxial loading and pure bending, and in beams under transverse loads. Using the methods of N. I. Muskhelishvili [Some basic problems of the mathematical theory of elasticity, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR 11, 626; 15, 370] he gives computed values for the direct stresses along the boundaries. *J. R. M. Radok.*

Lehnickiĭ, S. G. An approximate method of determination of the stresses in an elastic anisotropic plate near to an opening differing little from a circle. Inžen. Sb. 17, 3–28 (1953). (Russian)

The author uses conformal mapping and power series to find approximately stress distributions around holes mapped onto the unit circle by functions of the type

$$z=a\left\{\zeta+\epsilon\sum_{n=1}^{N}(\alpha_n+i\beta_n)\zeta^{-n}\right\}.$$

Instead of one boundary-value problem for the unit circle, he obtains two boundary-value problems for "unit ellipses", resulting from affine transformations of the unit circle. These transformations arise from the use of three complex variables

$$z=x+iy,\quad z_1=z+\lambda_1\bar{z},\quad z_2=z+\lambda_2\bar{z},$$

where λ_1, λ_2 are determined from the generalized biharmonic equation governing the plane problems of anisotropic elasticity. Several cases of almost square holes (with four axes of symmetry) under different external loading conditions are treated in detail and numerical results illustrating the method are given. *J. R. M. Radok* (Melbourne).

Irving, J., and Mullineux, N. Rectangular plates with stringers and ribs. J. Aero. Sci. 21, 847–848 (1954).

The authors use the Dirac delta function and finite Fourier transforms to find solutions of the differential equation governing deflections of thin plates subject to edge and lateral line loading. Their approach is proposed as an alternative to the Fourier-series treatment in a paper by the reviewer [same J. 21, 109–116 (1954); MR 15, 664] a major part of which is devoted to justifying the use of solutions of the type used by the authors. Otherwise there is little difference in the two approaches, both of which permit useful physical interpretation. *J. R. M. Radok.*

Poplavskiĭ, R. P. On a method of computing plates and membranes. The method of separation of the principal part of a solution. Inžen. Sb. 16, 149–172 (1953). (Russian)

The author defines as principal part of a solution in this context polynomial terms of the form $\Phi^{(i)}(x, y)F^{(i)}(x, y)$ for Poisson's and the nonhomogeneous biharmonic equations, respectively. Here $F^{(i)}(x, y)=f^i(x, y)-1$, with $f(x, y)=1$ being the equation of part of the boundary, expressible in polynomial form. Methods are given for determining the functions $\Phi^{(i)}$. They are demonstrated by application to elliptic, annular, semi-circular, polygonal, etc., regions subject to polynomial-type loading functions.

J. R. M. Radok (Melbourne).

Grigolyuk, È. I. Thin bimetallic shells and plates. Inžen. Sb. 17, 69–120 (1953). (Russian)

The author presents a general theory of bimetallic shells. A bimetallic shell consists of two layers, each having different mechanical properties. The surfaces of contact are rigidly united, they cannot slide against each other, they must move and be deformed together. The author modifies the Love-Kirchhoff hypothesis when applying it to two-layered shells. He uses the surface of contact as the initial surface and in his theory a homogeneous shell becomes a special case.

This paper is a summary of a number of the author's previous publications, e.g. (1) Inžen. Sb. 16, 119–148 (1953) [MR 16, 197]; (2) ibid. 7, 69–90 (1950); (3) ibid. 9, 177–200 (1951). The author draws attention to a mistake which he made in (3) and corrects it in this paper. He also shows an error in W. H. Wittrick, D. M. Myers, W. R. Blunden, Quart. J. Mech. Appl. Math. 6, 15–31 (1953) [MR 14, 928].

T. Leser (Lexington, Ky.).

Grigolyuk, È. I. Equations of axially symmetric bimetallic elastic shells. Inžen. Sb. 18, 89–98 (1954). (Russian)

This work is a continuation of the author's previous investigations on bimetallic shells reported in: (1) the paper reviewed above and (2) Inžen. Sb. 16, 119–148 (1953) [MR 16, 197].

The author derives the differential equations for thin-walled axially symmetric shells subjected to arbitrary loads and heating when displacements and deformations are small. A sphere, torus, cone, cylinder and circular plate are special cases of this general theory. Circular plates were investigated in (1), therefore here they are only briefly discussed. A cylinder, though analysed in (2), is given here a full treatment again because the equations in (2) were arrived at in a different way and the author wants to show that the results in both papers coincide. Conical and spherical shells are discussed in some detail, but the case of toroidal shells is only outlined. The author proves that if the differential equations for a given homogeneous shell are solvable, they are also solvable for a bimetallic shell of the same shape if Poisson ratios of both layers are equal.

T. Leser (Lexington, Ky.).

Alumyaé, N. A. On the determination of the state of equilibrium of a circular shell under an axially symmetric load. Prikl. Mat. Meh. 17, 517–528 (1953). (Russian)

When a circular cylindrical shell is subjected to axisymmetric pressure and axial stress, the axial symmetry of the deformation which obtains for small loading is known to fail and to become non-axisymmetric when the loading is sufficiently increased. The author considers the loading to increase proportionally to a parameter σ and then shows that, with suitable assumptions, the critical values of σ may be obtained from the stationary value of one integral and that the post-critical state of equilibrium may be characterised by the stationary value of another integral.

L. M. Milne-Thomson (Greenwich).

Mikeladze, M. Š. On the bearing capacity of initially anisotropic shells. Dokl. Akad. Nauk SSSR (N.S.) 98, 921–923 (1954). (Russian)

Forces in an orthotropic shell element are found in terms of the extension and curvature rates for the special cases $\epsilon_{12}=\kappa_2=0$ and $\kappa_{12}=\epsilon_2=0$. The analysis is based on the strain increment relations analogous to those of Levy-Mises for an isotropic material [R. Hill, Proc. Roy. Soc. London. Ser. A. 193, 281–297 (1948); MR 10, 83] and the usual Kirchhoff postulates. *R. M. Haythornthwaite.*

Iwiński, Tadeusz, and Nowiński, Jerzy. The transforms of differential equations of the theory of structures. Arch. Mech. Stos. 6, 343–362 (1954). (Polish. Russian and English summaries)

The authors propose that in statics, where a concentrated force is considered as the limit of a uniform load and a concentrated moment as the limit moment of a couple, the transition to the limits be done in the Laplace-transformed equations rather than in the original formulation of the physical problem. The proposed procedure is illustrated by several examples of bending problems. *M. Golomb.*

Filin, A. P. On a possibility of applying variational methods to structural mechanics. Inžen. Sb. 19, 125–140 (1954). (Russian)

Bodner, Sol R. The post buckling behavior of a clamped circular plate. Quart. Appl. Math. 12, 397–401 (1955).

A solution is given of the von Kármán equations for the radially symmetric post-buckling behaviour of a clamped circular plate under uniform edge pressure. The analysis parallels that of Friedrichs and Stoker [Amer. J. Math. 63, 839–888 (1941); J. Appl. Mech. 9, A-7–A-14 (1942); MR 3, 223, 288] for the corresponding problem of simple edge support. Notably it is again found that there is a boundary-layer phenomenon, viz. at sufficiently high values of p/p_{cr} (where p and p_{cr} are the acting and critical values of applied pressure, respectively) the radial membrane stress changes over rapidly from compression near the edge to tension in the interior. The asymptotic value $(p/p_{cr}\rightarrow\infty)$ of p_0/p (where p_0 is membrane stress at centre) is 0.13, and this contrasts with the value 0.47 found by Friedrichs and Stoker. *H. G. Hopkins* (Fort Halstead).

Traill-Nash, R. W., and Collar, A. R. The effects of shear flexibility and rotatory inertia on the bending vibrations of beams. Quart. J. Mech. Appl. Math. 6, 186–222 (1953).

Paper deals with the problem of lateral vibrations of an uniform beam when shear-flexibility (but not shear-lag) effects and rotatory inertia are taken into account. In the first part, the frequency equations are derived for the usual types of support. An interesting feature is the appearance of a series of high resonance frequencies, that has no counterpart in the classical theory and at which the contributions of bending and shear are roughly equal but in opposition.

In the second part, a vibration test on a free-free beam is described. Practical difficulties made impossible the excitation of frequencies from the new spectrum. The measured frequencies of the fundamental and the first overtone are in very good agreement with the calculated ones. Neglecting the rotatory inertia makes very little difference, but neglecting the shear stiffness gives serious errors. This is confirmed in part three, in which the effect of the shear flexibility on the bending vibrations of an airplane wing is considered.

W. H. Muller (Amsterdam).

Kruszewski, Edwin T., and Kordes, Eldon E. Torsional vibrations of hollow thin-walled cylindrical beams. NACA Tech. Note no. 3206, 33 pp. (1954).

The purpose of this paper is to consider the influence of so-called "secondary effects" upon the torsional vibration of hollow, thin-walled cylindrical beams. The effects considered are longitudinal (parallel to the axis of twist) inertia and longitudinal stress, the latter commonly referred to as the "bending stress due to torsion". The treatment is analogous to the analysis by Budiansky and Kruszewski [NACA Rep. no. 1129 (1953)] which took into account the influence upon transverse vibration of transverse shear deformation, shear lag, longitudinal and rotatory inertia. For torsional vibration, if x is the longitudinal coordinate and s the distance along the perimeter of the cross-section, then material displacement is assumed to be described by the two functions $\theta(x)$, representing cross-sectional rotation, and $u(x, s)$ representing longitudinal displacement. From these, expressions are found for the maximum strain and kinetic energies in vibration, and application of Hamilton's Principle leads to the appropriate differential equations and boundary conditions. In finding the general frequency equation for cylinders of uniform thickness, the variational formulation is used directly, with assumed Fourier series for the displacements. Numerical results are given for the free-free beam of rectangular cross-section in symmetrical vibration. Frequencies for the first four modes show that for medium and high aspect ratios, the secondary effects increase the frequency above that predicted by elementary theory, and that for this range of aspect ratios the effect of longitudinal inertia is negligible. For low aspect ratios, however, the torsional and longitudinal vibrations couple strongly, causing reduction of frequency in each mode below that of elementary theory. The present analysis for torsion deals with a more restrictive and idealized beam than for transverse vibration since doubly symmetrical cross-sections and closely spaced rigid bulkheads must be assumed. The results of this analysis are, moreover, intended for use chiefly as a check on the predictions of mathematically simpler models of thin-walled hollow beams. In a comparison with calculations on an equivalent four-flange box beam it is shown that the lowest three frequencies for this simplified model are in close agreement (but on the high side) with the more refined theory for aspect ratios greater than 2.

W. Nachbar (Seattle, Wash.).

Matuzawa, Takeo, und Hasegawa, Hirosi. Feldtheorie der Erdbeben: Elliptisches Quellengebiet. Bull. Earthquake Res. Inst. Tokyo **32**, 231–246 (1954). (Japanese summary)

The boundary conditions for an elliptic plate are formulated as follows: $p_{zz}=p_{zx}=p_{zy}=0$ at $z=-h$; $p_{zz}=-p$, $p_{zx}=p_{zy}=0$ at $z=h$ and $u=v=w=0$, $\partial w/\partial n=0$ at $z=0$ and $x^2/a^2+y^2/b^2=1$. Using Love's method for this problem the static displacements are obtained in the form of polynomials in x, y, z. The horizontal shear at the free surface as well as the stresses at the bottom of the plate are estimated for $b=2h$, $a=4h$, $\sigma=1/4$ and the plate is considered as an elliptic source of an earthquake. *W. S. Jardetzky.*

Matumoto, Tosimatu, and Satô, Yasuo. On the vibration of an elastic globe with one layer. The vibration of the first class. Bull. Earthquake Res. Inst. Tokyo **32**, 247–258 (1954). (Japanese summary)

Using Sezawa's form of solutions in the problem of vibrations of an elastic sphere [same Bull. **10**, 299–334 (1932)], the change of the period caused by a variation of elastic constants and density is investigated. The transverse vibrations are considered in the layered earth under assumption of a rigid or liquid core and of a solid mantle. In these two cases the period equation takes a simpler form and some of its roots are evaluated. *W. S. Jardetzky.*

Olszak, Wacław. Sur les bases de la théorie des corps élastoplastiques non-homogènes. I. Arch. Mech. Stos. **6**, 493–532 (1954). (Polish. Russian and French summaries)

The author remarks that the well known yield criteria and stress-strain relations for ideal isotropic elastic-plastic solids do not require homogeneity of the material. Final results of analysis of a thick cylinder that is nonhomogeneous radially are described: details are promised in part II. The author contends that deliberate introduction of nonhomogeneity will lead to economies in design.

R. M. Haythornthwaite (Providence, R. I.).

Lepik, Yu. R. Stability of a rectangular elastic-plastic plate nonuniformly compressed in one direction. Inžen. Sb. **18**, 161–164 (1954). (Russian)

The problem of the title is treated for simply-supported plates. The analysis only differs from that for elastic plates through the use of plastic moduli. Numerical results for a range of plate aspect ratios are given for the case when the applied end load varies linearly from zero across the plate width. These results are computed for plate material with specified values of the plastic moduli. The results need careful interpretation in view of recent work [e.g. H. G. Hopkins, Quart. Appl. Math. **11**, 185–200 (1953); MR **14**, 930; E. T. Onat and D. C. Drucker, J. Aero. Sci. **20**, 181–186 (1953); MR **14**, 929] illuminating the complex behaviour of plastic systems at instability. *H. G. Hopkins.*

Popov, S. M. Stability beyond the elastic limit of rectangular plates with off-center tension or compression. Inžen. Sb. **18**, 165–173 (1954). (Russian)

The problem of the title is treated for simply-supported plates, numerical results being computed for certain special cases. The analysis is similar to that in the paper reviewed above, and the reviewer's remarks on this latter paper again apply. *H. G. Hopkins* (Fort Halstead).

Matschinski, Matthias. Solution générale approximative des équations de la plasticité pour le cas de deux dimensions. C. R. Acad. Sci. Paris **239**, 1348–1350 (1954).

If a perfectly plastic material in plane strain satisfies Tresca's (or Mises', although the author does not mention this) yield condition, the Airy stress function must satisfy the non-linear relation $(\Phi_{xx}-\Phi_{yy})^2+4(\Phi_{xy})^2=k^2$. The author states that among all linear approximations to $(a^2+b^2)^{1/2}$, where $a<b$, the function $0.398a+0.960b$ is the best and uses this to approximate the yield condition. The resulting approximation contains eight possibilities depending upon the signs of $X_{11}-X_{22}$ and X_{12} and their relative magnitudes. In each case the author can write down the most general stress solution as one containing two arbitrary functions of one variable each. The author applies his approximation to an example involving the initiation of compressive buckling and predicts certain features of the resulting stress distribution. In view of the numerous examples in plane plastic strain which have been solved by other techniques, it is unfortunate that the author did not discuss such an example so that his method could be evaluated both for exactness and for simplicity. *P. G. Hodge, Jr.* (Brooklyn, N. Y.).

Have

★**Gross, Bernhard. Mathematical structure of the theories of viscoelasticity.** Actualités Sci. Ind., no. 1190. Hermann & Cie, Paris, 1953. 74 pp. 600 francs.

In this book are gathered together and correlated the many types of relations between stress and strain for viscoelastic materials. The treatment is restricted to cases in which only a single component of stress and of strain appear, so that scalar rather than tensor relations are discussed. Linear materials are considered and the development is based on the superposition principle, leading to mathematical relations between the creep function, relaxation function, retardation spectrum, relaxation spectrum, complex modulus and compliance. A table gives the structure of the relations between these several methods of expressing visco-elastic properties, delineating algebraic relations and the types of transforms involved. Both continuous and discrete spectra are considered, and exact and approximate methods of treatment. The treatment is integrated with related work on dielectrics and electric circuits, and has a bibliography of over eighty references. Particular applications of the theory are presented. This book provides a valuable integration of work otherwise spread through the literature. *E. H. Lee* (Providence, R. I.).

MATHEMATICAL PHYSICS

Grawert, Gerald. Eine Theorie der physikalischen Aussagen. Z. Physik **136**, 206–220 (1953).

Einige (zum Teil rein mathematische) Arbeiten zusammenfassend, werden Postulate für eine Theorie der physikalischen Aussagen mit eindeutiger normierter a-priori-Wahrscheinlichkeit angegeben. Durch Einführung einer relativen Messwahrscheinlichkeit wird diese Theorie weiter ausgebaut. Die gewonnenen Sätze entsprechen zwar keiner der bislang bekannten physikalischen Theorien, stehen jedoch der Quantentheorie äusserst nahe. (Author's abstract.)

C. C. Torrance (Monterey, Calif.).

Renaud, Paul, Joly, Maurice, et Dervichian, Dikran G. Organisation du voisinage G_v d'une grandeur G en vue d'une mesure précise. C. R. Acad. Sci. Paris **238**, 1389–1390 (1954).

Toute grandeur expérimentale G se discerne mal de son voisinage G_v. L'organisation de ce voisinage permet la mesure précise de G. Cette organisation correspond à l'annulation de deux jeux cohérents de dérivées partielles, l'un dans le système contenant G et l'autre dans l'appareil de mesure. (Résumé de l'auteur.) *C. C. Torrance.*

Renaud, Paul, Joly, Maurice, et Dervichian, Dikran G. Recherche d'une définition générale de l'énergie et des paramètres d'extension et d'intensité. C. R. Acad. Sci. Paris **239**, 1603–1605 (1954).

On propose une représentation schématique générale des relations entre l'énergie et les paramètres d'extension et d'intensité. Elle permet d'obtenir simultanément pour ces trois grandeurs des définitions valables même pour d'éventuelles formes nouvelles d'énergie. (Résumé de l'auteur.)
C. C. Torrance (Monterey, Calif.).

Optics, Electromagnetic Theory

Lohmann, A. Ein neues Dualitätsprinzip in der Optik. Optik **11**, 478–488 (1954).

The author suggests that the reciprocal nature of Fourier transforms used in discussing Fraunhofer diffractions can lead to a very useful duality principle, the impact of which is not limited to optical problems. The application of this principle permits, for instance, the transformation of statements in phase-contrast theory into statements in interference microscopy, statements on plane waves to equally valid statements on light coming from a finite object point.
M. Herzberger (Rochester, N. Y.).

Have

★Herzberger, M. Image errors and diapoint errors. Studies in mathematics and mechanics presented to Richard von Mises, pp. 30–35. Academic Press Inc., New York, 1954. $9.00.

In an earlier paper, Herzberger [J. Opt. Soc. Amer. **38**, 736–738 (1948); MR **10**, 220] introduced the concept of the diapoint of an object point, and proposed an image error theory based on it. (A diapoint of an object point is defined as the intersection point of the image ray with the meridian plane.) In the present note some new formulae relating to this theory are given, and it is claimed that the analysis of the diapoint configuration is mathematically simpler than the analysis of the configuration of the intersection points of rays with planes perpendicular to the axis.

The author asserts (p. 35) that T. Smith [Trans. Opt. Soc., London **23**, 311–322 (1922)] was the first to investigate for finite aperture and field the change of image errors when the object and stop are moved. Actually, formulae for the changes of image errors when the stop is moved are already contained in the investigations which were carried out much earlier by K. Schwarzschild [Astr. Mitt. Sternwarte Göttingen **9** (1905) = Abh. Ges. Wiss. Göttingen. Math.-Phys. Kl. (N.F.) **4**, no. 1 (1905), pp. 18–19]. *E. Wolf.*

Ukita, Yukichi, and Tsujiuchi, Jumpei. On the wave aberrations of the decentred lens system with the finite aperture. J. Phys. Soc. Japan **9**, 602–604 (1954).

Considering an axis point of a rotational optical system which is slightly decentered, the emerging wave generally has no symmetry of rotation. There will be astigmatism on the axis and meridional and sagittal asymmetric errors as well as a new spherical aberration in the meridional direction.
M. Herzberger (Rochester, N. Y.).

Blaschke, W. S. S. Field aberrations in wide aperture optical systems. Proc. Phys. Soc. Sect. B. **67**, 801–810 (1954).

The sine condition gives the laws for imagery for finite aperture and infinitely small field. The author investigates the laws for imaging with finite aperture for the points of a field where second-order aberrations of the field are permitted. The author derives the path difference (characteristic function) for the case of corrected spherical aberration, stating that it can easily be extended to the general case. A short (and incomplete) discussion of the geometrical meaning of the aberration terms is given. The reviewer would like to draw the attention of the author to one of his earlier publications on the subject [Phys. Z. **31**, 805–806 (1930)]. *M. Herzberger* (Rochester, N. Y.).

Ingarden, R. S., and Ochman, H. Optimal optical systems. Bull. Acad. Polon. Sci. Cl. III. **2**, 271–276 (1954).

The author starts from Lord Rayleigh's question as to whether it was possible to improve the quality of the final diffraction image by the matching of nonvanishing aberrations. He defines a function which he calls the resolution characteristic of the optical system and a condition for what he calls uniform meridional resolution. He gives a formula for meridional and tangential quality (probably meant sagittal quality) and calls the optical instrument an optimal one if the optical quality has a maximum for a given wavelength aperture, focal length, field of view and object distance. Lord Rayleigh's problem about the possibility of matching errors is not answered in the paper.
M. Herzberger (Rochester, N. Y.).

Blanc-Lapierre, André, et Dumontet, Pierre. Sur la notion de cohérence en optique. C. R. Acad. Sci. Paris **238**, 1005–1007 (1954).

In calculating the intensity distribution in an image plane produced by an optical instrument from a distribution of sources in the object plane or from luminous objects, one must take into account the distribution of phases of the sources of the object. In this note the author establishes a relation for the so-called coherence function (partial coherence factor) and the source distribution function on the basis of the theory of random variables. The phenomena of complete coherence, partial coherence and total incoherence are interpreted in terms of expectation values of the elementary sources. *N. Chako* (New York, N. Y.).

Godfrey, G. H. Optical diffraction effects produced by amplitude and phase changes in the wave front. Austral. J. Phys. **7**, 389–399 (1954).

Formulae are derived for the diffraction pattern of a light wave which is incident on a screen containing areas of any prescribed transmission properties, in terms of functions which characterise the diffraction by a screen with areas of the same shape and position, but of perfect transmittivity. The results, which are essentially a generalisation of Babinet's Principle are illustrated by applications to phase gratings, haloes, zone plates, phase-contrast techniques, holograms etc. The treatment is based on the elementary Kirchhoff scalar diffraction theory. *E. Wolf.*

Bramley, E. N. The diffraction of waves by an irregular refracting medium. Proc. Roy. Soc. London. Ser. A. **225**, 515–518 (1954).

In this paper it is shown that the results recently derived by Fejer [same Proc. **220**, 455–471 (1953); MR **15**, 761] on the diffraction of waves in passing through an irregular refracting medium can be deduced more simply by evaluating the statistics of the phase irregularities in the wave-front after traversing the medium and treating these irregularities as having been produced by a thin phase-changing screen. In establishing the equivalence of the two methods the author also shows that the angular spectrum as derived from considerations of multiple scattering, for transmission normally through a thick layer in which the dielectric constant is subject to random fluctuations can equally well be evaluated by considering an equivalent thin phase-changing screen. *S. Chandrasekhar* (Williams Bay, Wis.).

***Deppermann, Karl. Die Beugung von Planwellen an einer Kugel unter Berücksichtigung der Kriechwellen.** Dissertationen der Mathematisch-Naturwissenschaftlichen Fakultät der Westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 5, pp. 7–8. Aschendorffsche Verlagsbuchhandlung, Münster, 1954. DM 3.50.

See Deppermann and Franz, Ann. Physik (6) **10**, 361–373 (1952); MR **14**, 518.

Kay, Irvin, and Keller, Joseph B. Asymptotic evaluation of the field at a caustic. Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-55**, i+21 pp. (1953).

In this paper the authors have obtained an expression of the leading term of the asymptotic field when a plane normal incident wave is reflected from a parabolic and a circular cylinder, which is valid throughout the reflected space. Instead of calculating the exact field E which satisfies the Helmholtz equation, the radiation condition and the boundary condition $E=0$ of the surface Σ of the reflector (in this case Σ reduces to a curve Γ), the authors propose to determine the asymptotic form of the reflected field E_r satisfying the above conditions. The asymptotic form (a.f.) of E_r is determined from the knowledge of the a.f. of the incident field and from the boundary condition, the conservation of energy and the eiconal equation. This quasi-geometrical derivation of E_r is not valid on caustics. To overcome this difficulty the problem is formulated in terms of Green's theorem, where the field is given by

$$\text{(a)}\qquad E(x,y)=E_{\text{inc}}(x,y)+\frac{1}{4i}\int_\Gamma \frac{\partial E}{\partial n}H_0{}^1(kd)\,ds,$$

d being the distance from a point on Γ to (x, y). By substituting the asymptotic form of $\partial E/\partial n$ and of $H_0{}^1$ in (a), the reflected field is reduced to an integral over Γ which is evaluated by the stationary-phase method. This procedure yields a finite value for E_r on caustics. The leading term of E_r due to an nth order stationary point is found to depend upon k like $k^{\frac{1}{2}-1/n}$, $n>2$, provided the amplitude of the field is regular in the reflected space. For the parabolic cylinder, the leading term of E_r at the caustic (focus) contains $k^{1/2}$ as a factor and for the circular cylinder the dependence upon k is $k^{1/6}$. At the cusp the leading term is proportional to $k^{1/4}$. To determine the transition form of the field near the caustic and the cusp, the phase function is expanded up to fourth-order terms around the stationary point. Near the caustic the reflected field is then expressed by an Airy-Hardy function of third order, whereas in the neighborhood of the cusp it is given by a fourth-order Airy-Hardy function, provided the observation point (Aufpunkt) lies on the symmetry axis of the reflector. Two important results can be drawn from this analysis: (a) the validity of the Debye-Picht-Luneberg formulas which were derived on the basis of the asymptotic behavior of the field without taking into consideration the boundary values of E at the reflecting surface; and (b) the dependence of the leading term of the asymptotic field upon k can be assertained by examining the geometrical-optics field, i.e. by counting the number of rays passing through a point in the reflected space. This number gives the order of the singularity of the phase function (stationary point), from which one can immediately deduce the exponent of k in the leading term of the asymptotic expansion of the field, provided the amplitude is regular everywhere in the reflected region.

Apart from minor misprints the following errors have been observed. In eqs. (24) and (26) n should be in the denominator, if n is even, and $F_n=\exp[\pm\frac{1}{2}i\pi(1-n^{-1})\sin\pi/n]$ for sign $f^{(n)}(s)$ plus, respectively minus, if n is odd. In eqs. (44) and (48) $\cos(\pi/3)$ should be replaced by $\exp(-i\pi/3)\sin(\pi/3)$. The denominator in the argument of the cosine in eq. (55) belongs to the second term, but the final formula for the field given by eq. (56) is correct. Finally, in eqs. (59–62) α and β should be replaced by $k\alpha$ and $k\beta$. *N. Chako* (New York, N. Y.).

Schouten, J. P., and Beukelman, B. J. On the radiation pattern of a paraboloid of revolution. Appl. Sci. Res. B. **4**, 137–150 (1954).

This paper is concerned with the diffraction of an electromagnetic field by a finite section of a perfectly conducting paraboloid of revolution. The source of the field is assumed to be a dipole located at the focus, and it is assumed that the focal length is large compared to the wavelength. A Kirchhoff-like formula for the field in terms of the induced currents on the surface of the paraboloid of revolution is used as a basis for calculation; although the field in question is periodic, a time-dependent integral formula is used. The induced surface currents are replaced in the basic formula by currents which would come from an approximation of the incident field in the limit of small wavelength, and the resulting expression is evaluated in terms of Bessel and trigonometric functions. The far field is approximated and two curves showing the far field strength as a function of distance along the axis are given, each curve corresponding to a different orientation of the dipole. *I. W. Kay.*

Papas, Charles H. An application of Sommerfeld's complex order wave functions to antenna theory. J. Math. Phys. **33**, 269–275 (1954).

The wave functions mentioned in the title are obtained by separation of the scalar wave equation in spherical coordinates [Sommerfeld, Partial differential equations, Academic Press, New York, 1949, chapter V, appendix II; MR **10**, 608]. In the case of axial symmetry, the wave function expansion becomes

$$u(r,\theta)=\sum A_\nu h_\nu{}^{(1)}(kr)P_\nu(\cos\theta),$$

where the summation extends over complex values of ν. The system $h_\nu{}^{(1)}(kr)$ of spherical Hankel functions is complete and orthogonal for $ka\leq kr<\infty$; the coefficients A_ν are determined by the boundary conditions at $r=a$. The author

applies these functions to the boss antenna, consisting of a coaxial line fitted with an infinite flange and a hemispherical boss at the end of the inner conductor, and excited by the principal mode. By means of integral-equation and variational formulations, the author obtains a formal expression for the admittance of the antenna. *C. J. Bouwkamp.*

Chatterjee, S. K. Propagation of microwave through an imperfectly conducting cylindrical guide filled with an imperfect dielectric. J. Indian Inst. Sci. Sect. B. **37**, 1–9 (1955).

Chambers, Ll. G. An approximate method for the calculation of propagation constants for inhomogeneously filled wave-guides. Quart. J. Mech. Appl. Math. **7**, 299–316 (1954).

In a sense, the author reviews the various methods available for the mathematical description of waveguide propagation in cylindrical structures composed of contiguous regions of constant dielectric and magnetic permeabilities. A variational method is developed for the calculation of the propagation constant in such systems for the simplest cases. Conditions for the existence of transverse electric or magnetic waves are also considered. A numerical example is worked out. *C. J. Bouwkamp* (Eindhoven).

Freud, Géza. Über die Stromverdrängung in Leitern mit kreisförmigem Querschnitt. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 467–478 (1954). (Hungarian. Russian and German summaries)

The author computes the skin effect in a cylindrical conductor placed in a homogeneous sinusoidal magnetic field. He also computes the heat generated in the conductor. *A. Erdélyi* (Pasadena, Calif.).

Freud, Géza, und Szilvay, Gézáné. Über das magnetische Feld einer Parallelleitung. II. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 479–488 (1954). (Hungarian. Russian and German summaries)

Continuation of an earlier part [Freud, same Közl. **1**, 377–387 (1953); MR **15**, 272]. The inductance of the system is computed approximately under the assumption that the distance of the conductors is large or the frequency is small. *A. Erdélyi* (Pasadena, Calif.).

Marziani, Marziano. Sulla propagazione del fronte d'onda nei mezzi dispersivi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 683–687 (1952).

Starting with the assumption that the electric displacement $D(t)$ and the current density $u(t)$ could be expressed in terms of the electric field vector $E(t)$ in the form

$$(1)\quad \begin{aligned} D(t) &= \epsilon E(t) + \int_0^t \varphi(t-\tau)E(\tau)d\tau, \\ u(t) &= \sigma E(t) + \int_0^t \psi(t-\tau)E(\tau)d\tau, \end{aligned}$$

where $E(t)=0$ when $t<0$, the author has derived by means of Laplace transformation the following formula for the propagation of the electric field in a homogeneous medium in one dimension

$$(2)\quad E(z,t) \exp\left(-\frac{b}{c}z\right)E_0\left(t-\frac{z}{c}\right)+G\left(z,t-\frac{z}{c}\right)E_0(\ )d$$

by solving the one-dimensional Maxwell equations with the following boundary values: $E(z,0)=H(z,0)=0$ for $z>0$; $E(0,t)=E_0(t)$ for $t>0$ and $=0$ for $t<0$, $\lim_{z\to\infty}(Ez,t)=0$ for all $t>0$. In (1) ϵ and σ are the dielectric constant and the conductivity of the medium, φ, ψ are continuous functions of $t-\tau$ and possess a continuous first derivative. Equation (2) is derived on the bases that E, H and their first derivatives with respect to t admit a Laplace transformation in t and $g(t)=\varphi'+\psi(t)$ (kernel function associated with the law of dispersion of the waves in the medium) is at most an entire transcendental function of exponential type, and $G(z,t)$ is continuous in z and t. The velocity of propagation of the wave fronts is $c=(\mu\epsilon)^{-1/2}$ and $b=\gamma/2\epsilon$. From (2) one deduces that the velocity of the wave fronts along z remains constant as for non-dispersive media. Furthermore, it is shown that in the neighborhood of the wave front the electromagnetic fields do not vanish. *N. Chako* (New York, N. Y.).

Marziani, Marziano. Sulla propagazione del fronte d'onda nei mezzi dispersivi eterogenei. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 127–131 (1952).

The present note generalizes the results of the preceding paper when the dielectric constant ϵ and the conductivity γ of the medium are functions of z. By means of Laplace transformation the author obtains the following expression for the electric field

$$\text{(A)}\quad E(z,t)=\left(\frac{\epsilon(0)}{\epsilon(z)}\right)^{1/4}\exp\left\{-\int_0^z\frac{\gamma(z)}{2}\left(\frac{\mu}{\epsilon(z)}\right)^{1/2}dz\right\} \times\left[E_0(t-n)+\int_0^{t-n}E_0(\tau)G(z,t-\tau-n)d\tau\right],$$

where $n=\int_0^z(\mu\epsilon(z))^{-1/2}dz$. (A) is derived under the assumptions that $\epsilon(z)$, $\gamma(z)$ are bounded positive increasing functions of z in $0\leqq z\leqq d$ with bounded third-order derivatives. For $z>d$, $\epsilon(z)=\epsilon_0$ a constant, $\gamma(z)=0$ and $\epsilon_0<\epsilon(z)$. Furthermore, the fields E and H and the kernel $G(z,t)$ satisfy the same conditions as in the previous paper. Since n is an increasing function of z, the wave fronts are determined by setting $n=t$. The velocity of the wave front is given by $dz/dn=dz/dt=(\mu\epsilon(z))^{-1/2}$. In the vicinity of the wave front the propagation of the electric field is expressed by

$$\text{(B)}\quad E(z,t)\cong\left(\frac{\epsilon(0)}{\epsilon(z)}\right)^{1/4} \times\exp\left\{-\int_0^z\frac{\gamma(z)}{2}\left(\frac{\mu}{\epsilon(z)}\right)^{1/2}dz\right\}E_0(t-n)$$

since the second integral in (A) is negligible by taking n sufficiently close to t, and $E_0(+0)\neq 0$. One notices the resemblence of (B) to the expression obtained from the WKB method. The author points out that (B) cannot be derived from the theory of characteristics. *N. Chako.*

Bailey, V. A. Reflection of waves by an inhomogeneous medium. Phys. Rev. (2) **96**, 865–868 (1954).

A new approximate solution is given for the differential equation $d^2u/dx^2+p^2u=0$ where p is a given function of x, viz. $u_1=\exp\int^x q\,dx$, where $q=(p'/2p)[1-\{1-4p^4/p'^2\}^{1/2}]$ $(p'=dp/dx)$. The approximation to the second solution is then $u_2=u_1\int u_1^{-2}dx$. These approximate solutions are continuous at the zeros of p^2. The results are applied to finding the reflexion coefficient when plane waves in a uniform medium are incident normally on a stratified non-homogeneous layer and emerge into another uniform medium. *E. T. Copson* (St. Andrews).

De Socio, Marialuisa. Un teorema sul campo elettromagnetico. Boll. Un. Mat. Ital. (3) **7**, 423–427 (1952).

The author shows that if the electromagnetic fields are regular and at any instant are tangent to an open or closed surface S, then the energy flow across it vanishes always. The proof of this theorem runs as follows: Since E and H are tangential to S one can show that the fields E and H are derived from potential functions defined on S by simply integrating Maxwell equations around a closed curve C described on S. Application of Poynting's theorem results, after performing some vectorial manipulations, in the following expression of the energy flow:

$$\int_S (E\times H)\cdot n dS = -\int_C \Psi \text{ grad}_s\ \varphi\cdot t\ ds,$$

where $E=\text{grad}_S\ \varphi$, $H=\text{grad}_S\ \Psi$, and t is the tangent along C. Finally the author has drawn some conclusions with regard to the type of wave-guides within which electromagnetic fields can be propagated. *N. Chako* (New York, N. Y.).

Bron, O. B. The field as a form of matter. Èlektričestvo **1954**, no. 7, 3–10 (1954). (Russian)

An expository article on fields in which an attempt is made to fit the various concepts into the framework of dialectical materialism. *N. Rosen* (Haifa).

Durand, Emile. Les distributions de dipôles. Ann. Physique (12) **9**, 493–523 (1954).

The author shows that the relation $\mathbf{D}'=\epsilon_0\mathbf{E}+\mathbf{P}$ between the induction $\mathbf{D}'$, the field $\mathbf{E}$, and the volume density of polarization $\mathbf{P}$ holds equally well for distributions occupying zero volume if $\mathbf{P}$ is replaced by suitable functions of the type of the Dirac δ function.

It is shown that the two field vectors $\mathbf{E}$ and $\mathbf{D}'$ do not have the same sources. The vector $\mathbf{E}$ is due to fictitious charges equivalent to polarization. It is derived from a scalar potential, and its circuitation vanishes over a closed contour. The vector $\mathbf{D}'$ is due to fictitious magnetic currents equivalent to polarization. It is derived from a vector potential $\mathbf{A}'$, and its flux through a closed surface is always zero. In order not to camouflage these properties, it is important in the calculations to keep the discontinuous functions at the points of discontinuity. Numerous examples are given of distributions which exhibit these properties. An indication is also given as to how the results may be extended to magnetostatics. *J. E. Rosenthal* (Passaic, N. J.).

Durand, Emile. Sur la possibilité de considérer les potentiels et les champs comme des grandeurs densitaires; nouveau type de quadrivecteur. C. R. Acad. Sci. Paris **239**, 751–753 (1954).

This paper gives prominence to two retarded integrals in Maxwellian field theory. Consider, in flat space-time with $x_4=ict$, a field B^{pq} and current J^p satisfying $\partial_q B^{pq}=\mu_0 J^p$, $\partial_q \overline{B^{pq}}=0$, the overline indicating the adjoint or dual. Define Θ^{pq} by the volume integral

$$4\pi\Theta^{pq}=\int [B^{pq}]\frac{dv}{r}$$

([ ] indicates retarded value); then $\Box\Theta^{pq}=B^{pq}$. Define A^p and A_*^p by $A^p=\partial_q\Theta^{pq}$, $A_*^p=\partial_q\overline{\Theta^{pq}}$; then $\partial^pA^q-\partial^qA^p=B^{pq}$, so that A^p is the usual 4-potential (expressed as an integral in terms of current); the antipotential $A_*^p=0$. Define C^p by

$$4\pi C^p=\int [A^p]\frac{dv}{r};$$

then $\Theta^{pq}=\partial^pC^q-\partial^qC^p$. Θ^{pq} and C^p are found for an isolated moving charge and for a continuous distribution of current; in the latter case

$$C^p=\frac{\mu_0}{8\pi}\int R^pR_q[J^q]\frac{dv}{r},$$

$$\Theta^{pq}=-\frac{\mu_0}{8\pi}\int \{R^p[J^q]-R^q[J^p]\}\frac{dv}{r},$$

where R^p is the 4-vector drawn from the retarded event of integration to the event of evaluation. [Since $x_4=ict$, covariant and contravariant components are the same, and the author might have simplified his notation by using only subscripts.] *J. L. Synge* (Dublin).

Bastin, H., Hontoy, P., et Janssens, P. Sur l'application des méthodes topologiques de Poincaré au circuit non linéaire de Fruehauf. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 1199–1208 (1954).

Quantum Mechanics

Omnès, Roland. Le principe de Feynman en mécanique quantique non relativiste. C. R. Acad. Sci. Paris **240**, 497–499 (1955).

Novožilov, Yu. V. Causal operators in quantum field theory. Dokl. Akad. Nauk SSSR (N.S.) **99**, 533–536 (1954). (Russian)

A causal operator is defined as follows. Let (x) be a field operator of the usual kind in any quantum field theory. Let $|t_1\rangle$ and $|t_2\rangle$ be any two states of the system of fields, defined at two different times t_1 and t_2. Let t be the time of the field-point x. Then the causal operator $\chi(x)$ is defined by the equations

$$\langle t_2|\chi(x)|t_1\rangle=\langle t_2|\psi(x)|t_1\rangle,\quad t_2>t>t_1,$$
$$\langle t_2|\chi(x)|t_1\rangle=\langle t_1|\psi(x)|t_2\rangle,\quad t_2<t<t_1.$$

Some simple consequences of this definition are deduced. It appears to the reviewer that the definition is not self-consistent, because the states $|t_1\rangle$ and $|t_2\rangle$ might be chosen to be independent of t_1 and t_2 (for example by working in the Heisenberg representation). At least it is clear that $\chi(x)$ does not have the properties one usually associates with the word "operator". *F. J. Dyson* (Princeton, N. J.).

González Domínguez, A. Les parties finies des intégrales de Riemann-Weyl et les procédés de régularisation. C. R. Acad. Sci. Paris **240**, 499–500 (1955).

On donne une formule de régularisation pour les divergences ultraviolettes qui fait usage essentiel du concept de partie finie à l'infini d'une intégrale de Riemann-Weyl divergente. *Résumé de l'auteur.*

Polkinghorne, J. C. Renormalization of the transformation operators of quantum electrodynamics. Proc. Roy. Soc. London. Ser. A. **227**, 94–102 (1954).

Let the interaction between the fields in quantum electrodynamics be modified by multiplication with a function $V(x)$ of the space-time point x at which the interaction occurs. The S-matrix of the modified theory is then proved to be finite after renormalization, provided the variation of $V(x)$ is sufficiently smooth. The result is closely related

to the proof of finiteness of operators in the "Intermediate representation" defined by the reviewer [same Proc. 207, 395–401 (1951); MR 13, 608]. In particular, the author gives a simple motivation for the complicated definition of the transformation used by the reviewer. In general, the author's argument is much simpler and more direct than the reviewer's. *F. J. Dyson* (Princeton, N. J.).

Galanin, A. D., Ioffe, B. L., and Pomerančuk, I. Ya. Renormalization of mass and charge in the covariant equations of quantum field theory. Dokl. Akad. Nauk SSSR (N.S.) 98, 361–364 (1954). (Russian)

The coupled integral equations for the Green's functions of a system of interacting meson and nucleon fields are written down in a concise explicit form. The equations are then transformed into equations involving only renormalized quantities. Emphasis is put upon the logical basis for the definitions of the renormalized quantities, following the ideas of G. Källèn [Danske Vid. Selsk. Mat.-Fys. Medd. 27, no. 12 (1953); MR 15, 79]. *F. J. Dyson.*

Skobelkin, V. I. On double ray-refraction in nonlinear electrodynamics. Ž. Eksper. Teoret. Fiz. 27, 677–689 (1954).

A general non-linear electrodynamics is defined by a Lagrangian density $L=L(\xi,\eta)$, where $8\pi\xi=E^2-B^2$, $\eta=(E\cdot B)^2$. The equations of propagation are derived which describe an infinitesimal wave-disturbance superimposed upon a constant field. In general there will be four different velocities of propagation for a plane wave with its wave-front parallel to a given plane, because there are two directions of propagation and two states of polarization. Thus the non-linear properties of the vacuum in general cause a double refraction of light passing into a region of constant electric or magnetic field. The author proves that the propagation-velocities are independent of polarization, and therefore the double refraction is absent only if L has one of two special forms: (i) $L=\xi$, and (ii) $L=(4\pi C)^{-1}[1-(1-8\pi C\xi-C^2\eta)^{1/2}]$. Case (i) is the linear electrodynamics of Maxwell, case (ii) is the non-linear theory of Born and Infeld [Proc. Roy. Soc. London. Ser. A. 144, 425–451 (1934)]. *F. J. Dyson.*

Úlehla, Ivan. Relativistic wave equations for particles with spin 3/2. Čehoslovack. Fiz. Ž. 4, 101–109 (1954). (Russian. English summary)

The algebra of the linearized wave equations

$$\beta^j\partial\phi/\partial x_j-i\mu\phi=0$$

describing particles of maximum spin 3/2 is developed in considerable detail. It is shown that the matrices β_j can be represented as matrix products of the form $\gamma_j\times\alpha_j$, where γ_j are the Dirac matrices; the matrices α_j are reducible; each sub-matrix (there may be six of them) contains as multiplicative factor a complex number which can be chosen arbitrarily. One of the sub-matrices is the unit matrix; if only the multiplicative factor of this sub-matrix is assumed to be different from zero, the formalism reduces to Dirac's theory of the particle of spin 1/2. In the general case, the wave functions have 24 components, and represent particles which may have spin 3/2 or 1/2, and four different mass values. The latter are given in the form μ/α_i, where α_i is a simple algebraic function of the multiplicative factors. The possibility of applying the formalism in the theory of the elementary particles, for instance for the description of μ-mesons and of excited states of nucleons, is discussed. *E. Gora* (Providence, R. I.).

Úlehla, Ivan. On the theory of the equations for particles with a single spin 3/2 and with a single proper mass. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 11–27 (1954). (Czech. Russian summary)

The author has developed a formalism for the treatment of particles with maximum spin 3/2 in a previous paper [see the preceding review]. The conditions are now specified and discussed which one has to impose upon the β-matrices if the wave equation is to describe particles of spin 3/2 and of a single mass value only. It is shown that one can distinguish α-matrices of the following types: $\alpha(3,2,1)$, $\alpha(3,2,2)$, $\alpha(3,1)$, $\alpha(3,2)$, $\alpha(3)$; the numbers in parenthesis indicate that sub-matrices of third, second, and first order are contained in the matrix α. A matrix of the type $\alpha(3,1)$ is given which fulfills the conditions for single spin 3/2 and single proper mass. The interaction with the electromagnetic field is introduced in the usual way by adding the term $e\phi_i$ to the operator $i\partial/\partial x_i$. For the gyromagnetic factor the value $g=2/3$ is obtained in non-relativistic approximation. It is suggested that the μ-meson could be a particle of this type. *E. Gora* (Providence, R. I.).

Potier, Robert. Sur le développement en intégrales de Fourier, des fonctions d'onde des corpuscules à spin. C. R. Acad. Sci. Paris 240, 281–283 (1955).

Ivanenko, D., and Brodskiĭ, A. Interaction of gravity with vacuum particles. Dokl. Akad. Nauk SSSR (N.S.) 92, 731–734 (1953). (Russian)

A treatment of the interaction of the gravitational field with the vacuum of scalar and pseudoscalar particles (mesons) is based on the quantum theory of gravitation of the first author [see, e.g., A. Sokolov and D. Ivanenko, Quantum theory of fields, Gostehizdat, Moscow-Leningrad, 1952, Part II, Section 5; MR 14, 1044]. An expression is given for the Lagrange function of the problem, and used to derive the equations of motion and the commutation relations. It is then shown that one can determine an action function W of the problem. Explicit calculations are carried out under the assumption that the gravitational field is so weak that it can be described by linear equations. An iteration procedure is used to determine the action function W_0 for the vacuum in second approximation, that is the approximation which can be obtained with linear field equations. The calculation of higher approximations would require the use of non-linear field equations. The results obtained can be used to calculate the probabilities of effects which are due to the polarization of the vacuum, for instance, the formation of pairs of particles. *E. Gora.*

Hohlov, Yu. K. Description of the interaction of a system of particles with an electromagnetic field. Ž. Eksper. Teoret. Fiz. 26, 576–584 (1954). (Russian)

In the Schroedinger equation for a many-particle system interacting with the electromagnetic field, the author introduces a particular gauge for the electromagnetic potentials and thus is able to express the Hamiltonian of the system directly in terms of the electromagnetic field components. When the latter are expanded in series of spherical harmonics, the Hamiltonian is expressible in terms of a set of electric and magnetic multipole moments characterizing the system. These moments also enter into the matrix elements for radiative transitions. [See the related work of J. G. Brennan and R. G. Sachs [Phys. Rev. (2) 88, 824–827 (1952)] which is criticized by the author.] *N. Rosen.*

Takahashi, Yasushi. On gauge invariance and the structure of elementary particles. Progr. Theoret. Phys. 11, 251–263 (1954).

The self-energy of the photon and the matrix element of the γ-decay process of the π^0 meson as given by the second-order perturbation calculation are not gauge invariant. They are shown to be expressible in terms of the vacuum expectation value of the energy-momentum density tensor of the vacuum particles. *N. Rosen* (Haifa).

Nishiyama, Toshiyuki. A hydrodynamical description of many Bose particle systems. Progr. Theoret. Phys. 12, 265–278 (1954).

The hydrodynamical equations given in previous papers [same journal 8, 655–668 (1952); 9, 245–267 (1953); MR 14, 710, 1048] as an approximate description for a system of bosons of high density are further discussed with the aim of deriving the properties of the energy spectrum proposed by Landau for liquid helium II, mainly the phonon and roton spectra. Although phonons and rotons are indeed introduced, little progress is made, as far as the reviewer can see, toward properly justifying their existence in terms of the elementary interactions between bosons.
L. Van Hove (Utrecht).

Nishijima, Kazuhiko. Many-body problem in quantum field theory. II. Progr. Theoret. Phys. 12, 279–310 (1954).

After comparing various formulations of quantum field theory, the author considers the problem of normalization of the Bethe-Salpeter wave functions and that of solving scattering problems which involve the formation or destruction of composite particles, such as the deuteron. Expectation values of observables are expressed in terms of B-S wave functions and the differences between S-matrices for dressed and bare particles are critically examined. The covariant and contravariant notation of Part I [same journal 10, 549–574 (1953); MR 15, 589] is abandoned.
H. C. Corben (Pittsburgh, Pa.).

McCarthy, I. E. Physical properties of particles obeying generalized statistics. Proc. Cambridge Philos. Soc. 51, 131–140 (1955).

In the generalized statistics of Green [Phys. Rev. (2) 90, 270–273 (1953); MR 14, 1046] the maximum number of spin-$\frac{1}{2}$ particles allowed in an eigenstate is some positive integer and for particles of integer spins the field consists of some number of groups each of which may contain any number of particles. It is shown here that a consistent physical interpretation of generalized statistics is possible only when particles are created in pairs. The Feynman probability amplitude for an interaction is the same in generalized Fermi statistics as for the usual case, so that generalized Fermions could not be distinguished from ordinary Fermions by scattering experiments. The behavior of a generalized Fermi gas is the same as that of an ordinary Fermi gas except for strong degeneration. *H. C. Corben.*

Mitter, H., und Urban, P. Zur Streuung schneller Elektronen. I. Elastische Streuung. Acta Physica Austriaca 7, 311–323 (1953).

The methods of Feynman and Dyson are used to calculate the scattering cross-section for Dirac electrons in the electrostatic field of a nucleus. The scattering potential is assumed to be of the form $Ze \exp(-br)/r$. The Born approximation of nth order is shown to be valid provided that the factor $(Z\alpha/\beta)^n \ll 1$, where $\alpha = e^2/hc$ is the fine-structure constant, and $\beta = v/c$. However, terms containing this factor do not appear for all n since some of the integrals vanish; for instance, there is no such term for $n=3$. The non-divergent contributions to the scattering cross-section of order $n=2, 3$, and 4 are worked out. Simple formulas are obtained for $n=2$ only, and for $n=2$ and 3 if $b=0$, that is for the Coulomb potential. Formulas for the evaluation of the integrals appearing in the other terms are given in an appendix. Radiative corrections and radiation losses are not taken into consideration. *E. Gora* (Providence, R. I.).

Glaser, Walter. Zur Begründung der wellenmechanischen Elektronentheorie. Z. Physik 139, 276–301 (1954).

A formulation is given for the Lorentz theory, which permits a wave-mechanical theory of the electron to be derived as a natural generalization of the Maxwell-Lorentz theory. The treatment is based on the assumption of a (hidden) magnetization of vacuum, which is determined so as not to appear in the Lorentz equations of motion. This magnetization is considered to be the true cause of electron spin. In one interpretation the theory in the field-free case is shown to be equivalent to the Dirac system of equations. However, it appears that by proceeding in a direction different from that of the conventional interpretation it is possible to arrive at an understanding of the inner structure of the electron. This approach leads to the following basic results: a finite value for the rest energy of the electron and an "electron radius" connected with it as well as the (anomalous) factor e/mc between the angular momentum and the magnetic moment. In this covariant formulation of the wave mechanics of the electron, the radius of the electron, the Bohr radius, the De Broglie and Compton wave-lengths all appear consistently as a "characteristic length" of a charge distribution, or more precisely as the inverse of a four-vector with the dimension of an inverse length. The theory predicts that, besides the magnetic moment due to spin, the electron will have a small additional magnetic moment equal to the product of the electron charge and the electron radius. *J. Rosenthal* (Passaic, N. J.).

Enatsu, Hiroshi. Mass spectrum of elementary particles. I. Eigenvalue problem in space-time. Progress Theoret. Physics 11, 125–142 (1954).

The purpose of the paper is to try to avoid divergent self-energies and to get the mass spectrum of elementary particles. A generalized wave equation is assumed with one time-like variable and four space-like variables. The self-energy of a particle is considered to arise from the presence of a self-potential depending only on the space-like variables. On the basis of a number of assumptions, some of which appear rather doubtful, a mass spectrum is obtained which is similar to that of Y. Nambu [same journal 7, 595–596 (1952)]. *N. Rosen* (Haifa).

Ahiezer, A. I. Diffracted radiation of photons by particles with spin 1/2. Doklady Akad. Nauk SSSR (N.S.) 94, 651–654 (1954). (Russian)

Cross-sections for the scattering of very fast charged Dirac particles by a nucleus are obtained. Elastic collisions and collisions with the production of photons are considered. The formulas are valid only when the angles which the directions of the outgoing Dirac particles and the outgoing photons make with the direction of the incoming particles are small. The main mathematical tool is an analogue of Kirchhoff's formula for Dirac particles. The result in the

case of emission of photons is a modification of that obtained by Landau and Pomerančuk [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **24**, p. 505ff. (1953)] for spinless particles. *A. J. Coleman* (Toronto, Ont.).

Galanin, A. D. On the expansion parameter in the pseudoscalar meson theory with pseudoscalar coupling. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **26**, 417–422 (1954). (Russian)

The renormalized contribution to the nucleonic Green's function and to Γ_5 in neutral pseudoscalar meson theory is calculated (in the limit of meson mass approaching zero and for large momentum) for the five graphs of lowest order. For one type of repeated graph the results are summed giving a result analogous to one of Edwards [Physical Rev. (2) **90**, 284–291 (1953); these Rev. **15**, 83] for charge-symmetric theory. *A. J. Coleman* (Toronto, Ont.).

McConnell, James. Production and annihilation of negative protons. III. Proc. Roy. Irish Acad. Sect. A. **56**, 45–65 (1954).

[For parts I–II see same Proc. **50**, 189–221 (1945); **51**, 173–190 (1947); these Rev. **7**, 272; **9**, 400.] The cross-sections for nuclear pair production are computed on the pseudo-scalar meson theory with pseudo-vector coupling, the pairs being produced by meson-nucleon or by proton-neutron collisions, each case being treated by perturbation theory and by the theory of damping. The nucleon field is resolved into a field of pseudo-scalar mesons by the Weizsäcker-Williams method. Similar perturbation calculations are performed for pseudo-scalar coupling on the charged meson theory and including a treatment of meson-meson collisions. The two-meson mean life-time of a slow negative proton in lead is found to be 2×10^{-11} sec (p.v.) or 2×10^{-9} sec (p.s.). *H. C. Corben* (Pittsburgh, Pa.).

Thermodynamics, Statistical Mechanics

***de Beaumont, H. Thermodynamique. Principes et méthodes de thermodynamique rationnelle en vue de l'application aux écoulements rapides des fluides compressibles et du tracé conforme des diffuseurs et tuyères propulsives. Fasc. I.** Editions Industrielles, Techniques et Littéraires, Paris, 1954. 47 pp.

The author develops the fundamental differential relations via Pfaffians, and the equation of state via specific heats. He deals briefly with thermodynamic systems subject to the reaction $nM \rightleftarrows M_n$, and treats in some detail the Joule-Thomson effect. *C. C. Torrance.*

Kuper, C. G. Note on Ehrenfest's equations. Proc. Cambridge Philos. Soc. **51**, 243–244 (1955).

It is shown that, with proper interpretation, the equations of Ehrenfest, $\Delta c_p/vT\Delta\alpha = (dp/dT)_\Gamma$, $\Delta\alpha/\Delta\kappa = (dp/dT)_\Gamma$, hold under all transitions other than those of the first order.

C. C. Torrance (Monterey, Calif.).

Popoff, Kyrille, Dimitroff, Emmanuel, et Dotcheff, Kyrille. Sur une propriété des intégrales d'un système d'équations différentielles de la thermodynamique des processus irréversibles. C. R. Acad. Sci. Paris **239**, 1361–1363 (1954).

***ter Haar, D. Elements of statistical mechanics.** Rinehart & Company, Inc., New York, 1954. xix+468 pp. $8.50.

This book is intended primarily as a text book for graduate students in physics and, at least for some parts of the book, the student would have to be at a rather advanced level. The author attempts to include in a single book a survey of both the fundamentals and some applications of statistical mechanics, a worthy goal in view of the emphasis in most existing books on either one or the other of these two phases of the subject.

The book is divided into four parts: A. statistics of independent particles, B. ensemble theory, C. applications and D. appendices. Section A (95 pages) gives descriptions of most of the more elementary applications of statistical mechanics. There seems to be no very clear pattern with the topics changing frequently from perfect gases to imperfect gases, from stationary behavior to time-dependent behavior and back. At no place in this first section is there any discussion of the motivation or justification of introducing statistical concepts (this does not appear until page 129). Most new topics start with an equation to be derived later in the book and end with a reference to some other part of the book for an interpretation of the consequences.

Section B (72 pages) includes a survey of the usual types of ensembles. The organization is much better than in section A although there are still several gaps in the logical development and some incorrect statements. (The free energy is given "apart possibly for an additive constant," which should read "additive multiple of temperature". This error appears several places.)

The statistical methods are applied in section C (160 pages) to the equation of state (quantum and classical derivation of the virial expansions), condensation phenomena, electron theory of metals, semi-conductors, cooperative phenomena, topics in nuclear physics, the origin of the chemical elements and the theory of rubber elasticity. The emphasis is definitely on applications that are of current physical interest and still somewhat speculative, with very few of the more classic applications.

The appendix is quite large (116) pages. About half of this is a survey of the history of the H-theorem and ergodic theorems, much of it based upon the Ehrenfest papers [P. and T. Ehrenfest, Encyk. Math. Wiss., Bd IV 2 II, H. 6, Teubner, Leipzig, 1912]. The reviewer thought this was by far the best part of the book even though the author is quite generous in his use of the work "proof." This does give a very clear picture of what people have tried to prove and why. There are shorter appendices on irreversible processes, the third law of thermodynamics, the Darwin-Fowler method, intermolecular forces and relativistic statistics.

The author has included very generous lists of references throughout the book. Because of this and the inclusion of several applications not found in other books, this book will probably have more appeal as a reference than as a text. The logical development suffers very much from the author's attempt to spare the reader from any consideration of ensemble theory until he has been exposed to most of the consequences of the theory. The setting aside of ergodic theory and the third law of thermodynamics in an appendix seems quite unjustified. *G. Newell* (Providence, R. I.).

Jauho, Pekka. A central theorem of statistical mechanics. Ann. Acad. Sci. Fenn. Ser. A. I. Math.-Phys. no. 172, 11 pp. (1954).

The theorem in question is the equivalence of the definitions of entropy given in thermodynamics and in statistical mechanics. The mathematical level of the derivations and the preciseness of the definitions are in keeping with those used in the studies of statistical thermodynamics.

G. Newell (Providence, R. I.).

Landsberg, P. T. Method of transition probabilities in quantum mechanics and quantum statistics. Phys. Rev. (2) 96, 1420–1427 (1954).

A mathematical analysis is given for the interrelations between a number of assumptions and theorems commonly used in quantum statistics. It extends previous work of J. S. Thomsen on the same subject [Phys. Rev. (2) 91, 1263–1266 (1953); MR 15, 491]. The various physical interpretations which can be given to the formal concepts and theorems are also discussed in some detail.

L. Van Hove (Utrecht).

Bayet, Michel, Delcroix, Jean-Loup, et Denisse, Jean-François. Théorie cinétique des plasmas homogènes faiblement ionisés. I. J. Phys. Radium (8) 15, 795–803 (1954).

A method is described for obtaining the distribution function for the charged particles of a weakly ionized homogeneous gas in both an alternating electric field and a static magnetic field. Equations are derived for the coefficients of the expansion of the distribution function in powers of the electric field strength. Each of these coefficients is then solved by means of a spherical harmonic expansion of the velocity dependence. The method is applied particularly to a perfect Lorentz gas in which only collisions between electrons and neutral particles are considered and energy exchange is neglected. The first two approximations are considered in detail, from which an expression is obtained for the conductivity tensor in the presence of a magnetic field.

G. Newell (Providence, R. I.).

Siegert, A. J. F. On the theory of condensation. Physical Rev. (2) 96, 243–249 (1954).

The object of this paper is to explain why approximations of the van der Waals type give results which are usually rather good quantitatively but give a qualitatively incorrect description of condensation itself.

The assumption is made that for submacroscopic cells, the probability (calculated in the usual way) that there will be n particles in a given cell is a function of n with two sharp maxima at n_1 and n_2. Here n_1 and n_2 are approximately the number of particles associated with the gas and liquid states. As long as the cells are finite in size and statistically independent of each other, a phase transition is impossible; one must have an interaction between cells. The interaction energy between cells is assumed to act only between nearest neighbors (surface interaction) and depend only upon the values of n for these cells. This energy depends primarily upon whether n is approximately n_1 or approximately n_2 and so, for this part of the calculation, the state of the cell has essentially just two possibilities, n_1 or n_2. The inclusion of an interaction of this type gives a problem mathematically equivalent to the Ising problem.

A true condensation obtains as a result of this interaction, corresponding to the transition at low temperatures of the magnetization at zero field in the Ising ferromagnet from $+I$ to $-I$. The Curie temperature of the Ising problem does not enter into the problem since the interaction energy between cells, even though it is weak, is still large compared with kT. The net effect of including the interaction between cells is to cause an actual condensation where otherwise there would be only a rapid continuous change of state. The explanations given in this paper may pave the way to a justifiable scheme of approximate analysis of phase transitions based upon the properties of finite and perhaps even small collections of particles and their energies of interaction.

G. Newell (Providence, R. I.).

Gold, Louis. Direct cellular evaluation of the density of states in phase space and the accurate calculation of Fermi levels. J. Appl. Phys. 25, 1278–1280 (1954).

The author claims to have developed "a new procedure for rigorously deducing" the number of points of a cubic lattice that lie inside a sphere of given radius, as used, for example, in computing the energy distribution of a quantum-mechanical ideal gas.

After introducing some confusing notation for counting points, he eventually resorts to approximating sums by integrals (as has always been done for large spheres) but manages to integrate twice over certain parts of this sphere and arrives at a final answer that is larger than the volume of the sphere by 6% for spheres of arbitrarily large radius. He concludes that the usual answer "compares quite favorably" with his. By the same new procedure, he does succeed in giving a "rigorous proof" that a cube of side n lattice spacings contains n^3 points.

G. Newell.